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BIOMETRIKA

A SECOND PIEBALD FAMILY FROM SUFFOLK.

BY E. A. COCKAYNE, D.M., F.R.C.P.

IN 1914 I published an account of a piebald family*, which originally came from the neighbourhood of Bury St Edmunds in Suffolk. About a year ago at the Hospital for Sick Children, Great Ormond Street, I saw two piebald children, members of a family whose ancestors lived near Ipswich in Suffolk. It has been impossible to trace any relationship between these two families, nor even to find a surname common to both. In both of them there has been descent through females and neither knows the name of the member who first came to London. It is, however, most improbable that two unrelated families with so rare a condition lived in the same county. The only other piebald family recorded in England is the London one described by Bishop Harman in 1909 and this may be descended from the same stock.

I am only able to give a short and incomplete pedigree, because the members of the family, whom I saw, have lost touch with their cousins and it has been impossible to communicate with them.

I, 2. Had a white frontal blaze.

II, 1. Has dark hair and is normal, and has eight or nine normal children.

II, 5. Has a white frontal blaze and is childless.

II, 9. Has a white frontal blaze, but all her four children are normal, and one of her daughters has a number of children and grandchildren, who are likewise normal.

II, 11. Piebald.

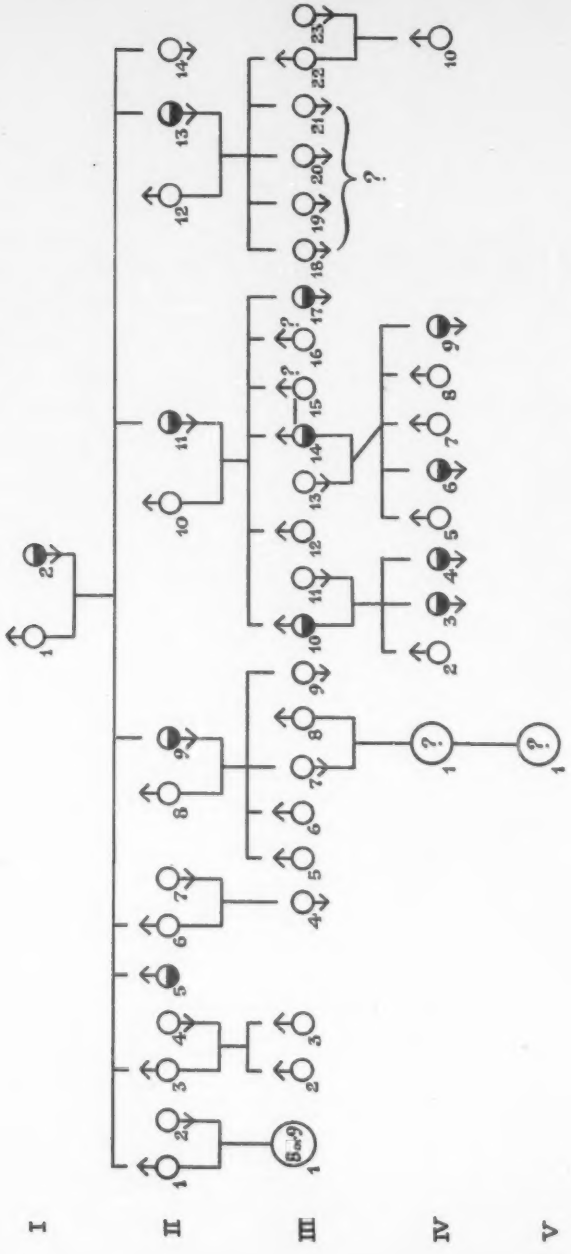
III, 10. Aged 43. Has very fair hair with distinct white frontal blaze of medium size. The inner third of the left eyebrow is white. Almost the whole of the skin on the front of the chest and the greater part of that on the abdomen is unpigmented and was severely reddened and blistered by the sun years ago, when he was in Malta and Corfu. Large areas of skin on the backs of both arms above the elbows and on the legs round the knees and upper part of the calves is also unpigmented, and there are smaller patches on the forearms and backs of the legs.

III, 12. Normal with rather dark brown hair.

IV, 2. Aged 18 years. Is dark haired and normal.

IV, 3. Aged 14 years. Has blue eyes and very fair hair. She has a large white frontal blaze and a triangular area of unpigmented skin on the forehead, which ends above the root of the nose. The whole of the skin on the posterior

* E. A. Cockayne, *Biometrika*, Vol. x. 1914, p. 197.


I
II
III
IV
V
?
?
?
Born?

surface of the lower two-thirds of both upper arms is unpigmented and ends sharply in a curved line with its convexity upwards. There are smaller and less clearly defined patches on the forearms. The skin over the knees and for a short distance above and a greater distance below them is also unpigmented, and forms a large oval well-defined area on the front of each leg. There are other similar patches on the upper part of the calves. The unpigmented skin becomes bright red when exposed to the sun, whereas the pigmented skin turns brown, but in winter little difference can be seen between them. A large part of the skin of the chest and abdomen appears to be unpigmented, but has never been exposed to sunlight.

IV, 4. Aged 9 years. Is very fair with blue eyes. She has a much smaller frontal blaze than her sister, but the V-shaped area of unpigmented skin on her forehead was very clearly defined, when I saw her, as it was still bright red from exposure to the sun. The unpigmented skin ends a short distance above the root of the nose and there are no hairs in her eyebrows or eyelashes. The skin over the elbows and on the backs of the arms above them is unpigmented and separated from the normal skin by a curved line, like that in her sister, but situated nearer to the elbow. The skin over the knees and above and below them is unpigmented, and these areas are similar in shape and size to those in her sister. There are other patches of unpigmented skin on the forearms and on the legs, especially on the upper part of the calves. The skin on the chest and abdomen has never been exposed to the sun, and it is impossible to be sure that any of it is unpigmented, though a large part of it appears to be.

III, 14. I have been unable to see this man, but a photograph I have seen shows a small white frontal blaze. His brother says that he has other unpigmented areas similar in distribution to his own. His wife, III, 13, has rather dark brown hair and brown eyes.

IV, 5. Aged 13 years. Is dark and normal.

IV, 6. Aged 10 years. Has grey eyes and fair hair with a very large white frontal blaze. The skin of the forehead and front of the neck is unpigmented. The greater part of the skin of the right arm over the deltoid, that of the inner side of the upper arm and along the radial side of the forearm almost as far as the wrist is unpigmented, and there is an area of unpigmented skin over the elbow. On the left arm the skin over the deltoid is unpigmented, and there is another unpigmented area along the radial side of the forearm.

IV, 7 and 8. Aged 5 and 3 years respectively. Grey eyes and light brown hair. Normal.

IV, 9. Aged 2 years. Very fair hair and grey eyes. There is a small white frontal blaze and the skin of the upper part of the forehead is unpigmented. On other parts of the body the skin is so fair that it is impossible to distinguish the unpigmented from the normal skin.

III, 15. The twin brother of III, 14 died at birth, and it is not known whether he was piebald or not.

III, 16. Died in infancy. It is not known whether he was piebald or normal.

III, 17. Piebald. Died unmarried.

II, 13. Piebald. The condition of her children, III, 18 to 22, and of her grandchild, IV, 10, is not known.

In connection with the family reported in 1914 I should like to state that another piebald boy, a younger brother of VI, 2, was born soon after the publication of the paper. Unlike his brother he had very dark brown hair and his frontal blaze was smaller. On p. 200 I said that it was probable that in some members of the family at least the areas of unpigmented skin were larger and more numerous than I had stated. Proof of this was obtained in the case of the boy VI, 2. I saw him after he had been bathing at the seaside and found that he had extensive areas of unpigmented skin on the chest, smaller ones on the abdomen and flanks, several spots of unpigmented skin on his penis, and large patches on the legs as well as on the arms, so that it is probable that other members of the family had white patches on the legs, though they said that this was not so. There is little doubt therefore that the distribution of unpigmented skin is very similar in these two families, as one would expect if they have a common origin, though none of the members of the family described in this paper has heterochromia iridis. They are like the majority of hereditary and isolated cases described by others in having the dorsal region fully pigmented and in possessing a white frontal blaze. Jenks has described a family of American whites of Welsh-Scottish origin*, in which the piebald pattern, transmitted by direct descent, differed very much from the usual one. All the piebald members had a stripe of unpigmented skin running down the mid-dorsal region and most of them had no frontal blaze in spite of having very extensive areas of unpigmented skin on the trunk and limbs.

In this family as in other published cases, with the exception of the two families recorded by Meirowsky and Spickernagel, and by Meirowsky respectively†, the condition has only been transmitted by those affected. Assuming that the condition is due to a single dominant gene, and that the piebald parents were all heterozygous, the expectation is that the piebald and normal members will be equal in number. In the family described in this paper in five complete sibships with 26 members, 11 were piebald, 13 were normal, and in 2 the condition is not known, and in the five sibships of the family described in 1914 with 28 members, 15 were piebald and 13 normal. The numbers agree closely with those expected, if the piebald condition is due to a dominant gene. If, however, the full sibships in this family and the ten others I collected in my book are taken together, there is a slight excess of piebalds, 144 piebalds and 131 normal members. The two exceptional families, in which normal individuals are said to have transmitted the condition, and the one described by Jenks, are omitted from these figures. Considering the sex incidence in the same eleven families, there were 79 males and 81 females with the piebald pattern, so that both sexes are about equally affected.

* Jenks, *American Anthropologist*, Vol. xvi. 1914, p. 221.

† E. A. Cockayne, *Inherited Abnormalities of the Skin and its Appendages*, 1933, p. 54. (Bibliography.)

IV, 9.

IV, 6.

III, 10.

IV, 3. IV, 4.

IV, 3. IV, 4.

PEDIGREES OF HEREDITARY EPICANTHUS.

By C. H. USHER, M.B.

In 1828 Schön* described the condition now known as epicanthus. Von Ammon in 1831 named it and in 1860 he classified it. His classification is given in my paper on a pedigree of epicanthus and ptosis in the *Annals of Eugenics*†. The epicanthic fold is represented in all races during foetal life‡. Waardenburg§ found epicanthus present from the beginning of the third to the beginning of the sixth month of pregnancy. In his embryos it was never present as a fold covering the canthus medialis. Eisler‖ states that the human foetus of 28–30 cm. has an epicanthus which coalesces only with the upper lid fold to form the typical mongolian fold. In the first six months of life, according to Martin's statistics¶ for epicanthus in the Munich population, 33·1 °/₀ of males and 32·6 °/₀ of females have epicanthus, but the numbers are reduced to 3·3 °/₀ and 2·6 °/₀ respectively in the years 12 to 25, thus showing that in most cases an epicanthus disappears in early life. In consequence peculiar difficulties are encountered in the investigation of a pedigree of simple epicanthus and these are added to by the circumstance that individuals who have epicanthus and their relations are usually unaware that an abnormal condition is present. This is in marked contrast with the characteristic cases of ptosis with epicanthus which are well known to the relatives. Clinical observation sometimes shows the presence of an epicanthus at the first examination and its absence at a subsequent examination. Thus: (1) A female (Fig. 21, II, 1, p. 15) at age of 5 years had well-marked bilateral epicanthus, which had disappeared by the age of 24. (2) Another female (Fig. 12, II, 8, p. 11) child when seen at 5 years of age had a unilateral epicanthus, which was not present 21 years later. (3) A male (Fig. 17, III, 1, p. 13) with bilateral epicanthus in infancy had at the age of 7 no epicanthus on right side, but the remains of one on left side. (4) A female when seen in infancy and again at 5 years of age had bilateral epicanthus, which was not present at the age of 9.

Published pedigrees of simple epicanthus are few. I know of ten of which Meirowsky's is the only large one, though in a pedigree by Rosenstein, epicanthus occurs in four generations. The first of these ten pedigrees to appear was V. Ammon's published in 1860.

* Schön, Matth. Joh. Albrecht, *Handbuch der pathologischen Anatomie des menschlichen Auges*, Hamburg, S. 59—60, 1828.

† *Annals of Eugenics*, Vol. I. Parts I and II. pp. 128.—138, 1925.

‡ Keith, *Human Embryology and Morphology*, 5th ed. p. 244, 1933.

§ Waardenburg, P. J., *Graefes Arch. f. Ophth.* S. 124, 221—229, 1930.

‖ Eisler, P., *Kurzes Handbuch der Ophthalmologie* (Schieck u. Brückner) I. S. 230, 1930.

¶ Martin, Rudolf, *Lehrbuch der Anthropologie*, Jena, S. 425, 1924.

V. AMMON, Fig. 1, observed a son and three daughters with a small tarsal epicanthus which they had inherited from their mother.

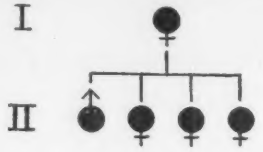


Fig. 1.

DREWS, Fig. 2, knew a family in Munich in which a mother, her son and daughter had epicanthus.

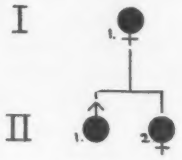


Fig. 2.

GALEZOWSKI, Fig. 3, in 1875 wrote that epicanthus is most frequently double, and it is congenital. He had seen it hereditary in a family; the father and the two children presented the same deformity in an excessive degree.

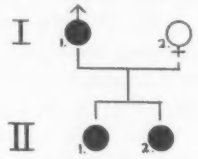


Fig. 3.

MANZ, Fig. 4, saw five children in a family of ten with epicanthus. The other five children did not have the deformity. Sex and order of birth not given. No mention of eye movements.

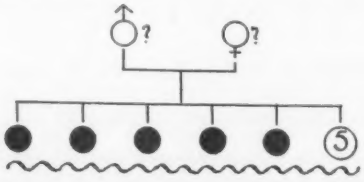


Fig. 4.

SCHEIDT, Fig. 5, observed partial lid fold and epicanthus in four of six children of a family of Alpine descent, whose parents, grandparents and brother and sister were not affected.

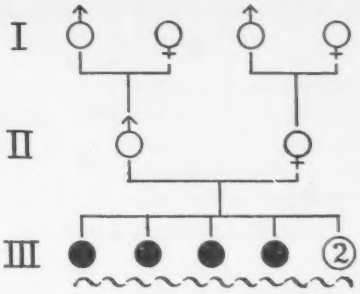


Fig. 5.

ROSENSTEIN, 1st family, Fig. 6, IV, 2, a girl, age 9, with bilateral epicanthus and strabismus, H. 6 D. in right and 4 D. in left eye and 1 D. of astigmatism in each. Asymmetry of face. IV, 1, her brother, age 13, with bilateral epicanthus and concomitant convergent strabismus. H. R. 3 D., L. 4 D. with 1 D. of astigmatism in each. No facial asymmetry. The mother, III, 1, age 43, has bilateral epicanthus,

Fig. 6.

H. 6 to 7 D., astigmatism 1·5 D. in each. Very marked facial asymmetry. II, 1, the maternal grandmother, always had bad vision, bridge of nose was flat with vertical skin fold on both sides. I, 1, the great grandmother, had always poor vision, also epicanthus and facial asymmetry. There is direct transmission of epicanthus in four generations.

MEIROWSKY, Fig. 7 (p. 8). In generation I a female married twice. From the first marriage she had a son and daughter both with epicanthus. The son had nine children, all males, of whom six had the hereditary character. All affected members

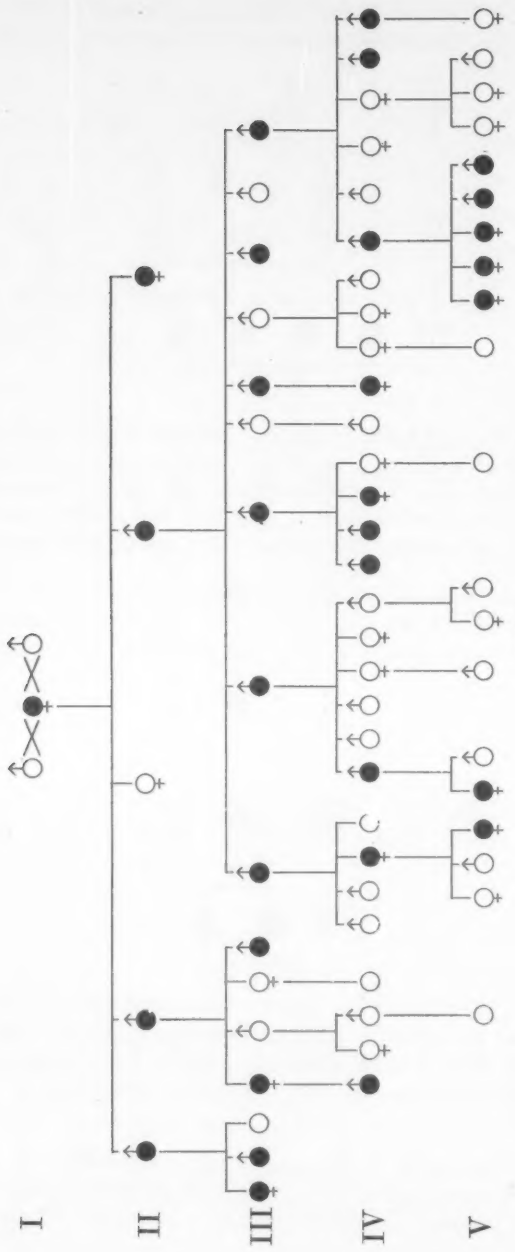


Fig. 7.

who married transmitted epicanthus to some of their descendants, whilst the unaffected members had children without epicanthus. In the same manner the three children of the second marriage of whom two were affected, both males, transmitted the affection to two further generations. Of the 87 descendants of the female in generation I, 31 have epicanthus.

ROSENSTEIN, 2nd family, Fig. 8, III, 1, a male, age 29, with bilateral epicanthus and a high degree of myopia, R. 13 and L. 16, astigmatism of 1 D. in each and blepharochalasis of left. III, 2, his twin brother, with a less degree of myopia, had the same condition. III, 3, the youngest brother, a 14-year-old boy, who died of inflammation of the lungs, was short-sighted and had exceptionally well-marked bilateral epicanthus. III, 4, two unaffected brothers. II, 1, the father, died. He had a high degree of short sight and bilateral epicanthus. It is not known whether any of the grandparents (I, 1—4) had a similar anomaly. There is direct inheritance of epicanthus with other anomalies of the eye from father to three sons.

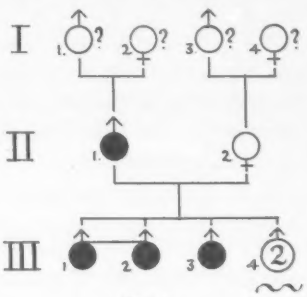


Fig. 8.

ROSENSTEIN, 3rd family, Fig. 9, III, 1, a 15-year-old boy with bilateral epicanthus and myopia, R. 5 D., L. 7 D., with astigmatism of 1 D. in each. He is an only child. II, 1, his father, died. He had bilateral epicanthus and very poor vision both far and near. It is not known whether the boy's grandparents (I, 1 and 2) or near relations had a similar anomaly. There is direct inheritance of the deformity from father to son.

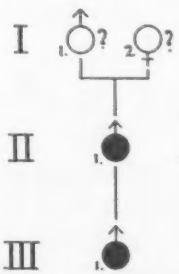


Fig. 9.

WAARDENBURG, Fig. 10, observed epicanthus in five brothers. Their father did not have the character, but their mother had it in a slight degree.

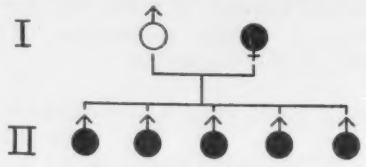


Fig. 10.

In these 10 pedigrees are 69 persons with epicanthus; 35 are males, 23 are females, and in 11 the sex is not recorded. In 8 pedigrees (Figs. 1—3, 6—10) the inheritance is continuous with transmission through the male thirteen times and through the female nine times. In 2 pedigrees (Figs. 4 and 5) epicanthus is confined to a single sibship. Consanguinity is not recorded in any of the pedigrees.

In submitting the results of an investigation of some new pedigrees of simple epicanthus—24 in all—I am fully aware how unsatisfactory these are in some of the pedigrees, owing to the probable disappearance of the epicanthus in certain cases, but I venture to record them in the hope that when considered collectively some light may be thrown on such pedigrees*.

My own material consists of epicanthus pedigrees which show heredity. In some of these, investigation originated from clinical notes made in some instances many years ago, and in others from cases which Professor Low† had seen from one to three days after birth in the years 1924—27. With few exceptions, only examined cases of epicanthus occurring in the pedigrees have been accepted. None of the pedigrees is large. Examples have already been given of the disappearance of epicanthus. In addition to these proved cases, in some pedigrees individuals without an epicanthus are suspected of formerly having had this character. The three youngest members of a sibship, in Fig. 11, have bilateral epicanthus, but

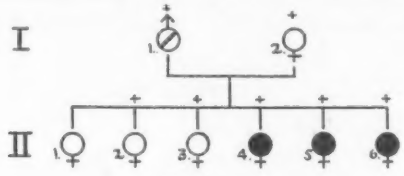


Fig. 11.

two older ones when they were examined had no epicanthus. The first-born, not seen, had died in infancy. The distribution of the abnormality is suggestive that

* Some of these pedigrees are briefly described on pp. 10—18, and fuller particulars of each individual pedigree are provided on pp. 18—25.

† Professor Alexander Low, University of Aberdeen.

one or more of the three first born sibs may have had epicanthus in infancy. Their father has congenital colobona of iris and microcornea.

(The + sign above individuals in the pedigree charts indicates that they have been examined.)

In Fig. 12 are two sisters with affected offspring. One of them, II, 8, had unilateral epicanthus at the age of 5 years, whereas there was no epicanthus

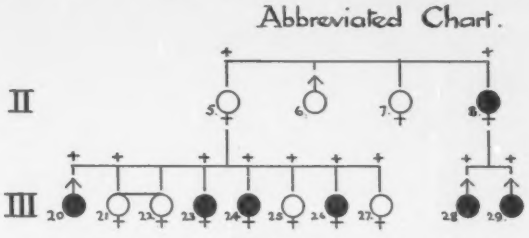


Fig. 12.

when she was 26 years of age. An older sister, II, 5, with four affected children, has no epicanthus. She had not been seen in childhood, and similarly with her mother (not shown in abbreviated chart), I, 2. It may also be of significance that only the youngest of the four affected children of II, 5 has bilateral epicanthus, in the others it is unilateral.

In Fig. 13, a mother with two children, all showing well-marked bilateral epicanthus, said that her mother, I, 1, had the same condition. Her medical man,

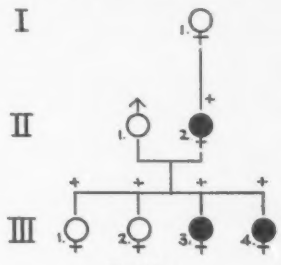


Fig. 13.

however, reported that she has no traces of epicanthus. Had it possibly disappeared?

In Fig. 14 are two sisters with affected offspring. One of them, II, 5, has a definite epicanthus on left side only. Her sister, II, 4, who is older than she is, has no epicanthus. Their parents, I, 1 and 2, have no epicanthus. The questions arise: first, had II, 5 bilateral epicanthus at an earlier age and, secondly, did II, 4 and I, 1 or I, 2 formerly have epicanthus?

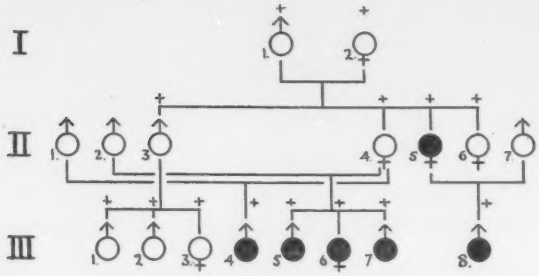


Fig. 14.

In Fig. 15 a male, IV, 3, with bilateral epicanthus in infancy and at age of 5 has now, at the age of 9, what appears to be only the upper part of each epicanthus remaining. The boy's maternal aunt, III, 13, age 12, has a bilateral fold resembling

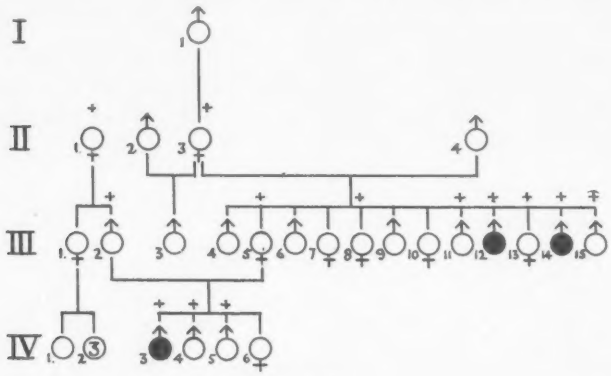


Fig. 15.

the upper part of an epicanthus, which does not extend down so far as the internal tarsal ligament. She is marked in the chart as having no epicanthus, though possibly, as in the case of IV, 3, a definite bilateral epicanthus was present earlier. These cases (IV, 3 and III, 13) and also the case of a male, II, 3 in Fig. 29, suggest that the epicanthus disappears from below upwards. In the mother's sibship, epicanthus occurs only amongst the younger members (Fig. 15, III, 12 and 14), so that the mother at their age may have had the same condition.

The next four pedigrees of epicanthus in three generations illustrate direct inheritance of epicanthus. Fig. 16, a male, I, 2, is dead, but his photograph shows epicanthus. He married twice and transmitted the condition to a member of both his families. His daughter, II, 2, and her son, III, 3, have epicanthus. It is unknown whether II, 1, 3 and 4, three unaffected parents who have children with epicanthus, had the same condition in childhood.

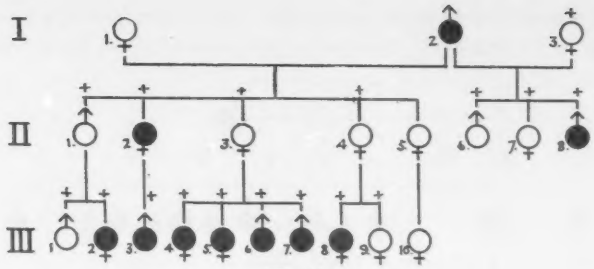


Fig. 16.

Fig. 17. Albert S., III, 1, age 7, with remains of an epicanthus on left side only, in infancy had bilateral epicanthus. His brother, III, 3, and sisters, III, 2 and 4, have bilateral epicanthus, which is most conspicuous in the youngest sister,

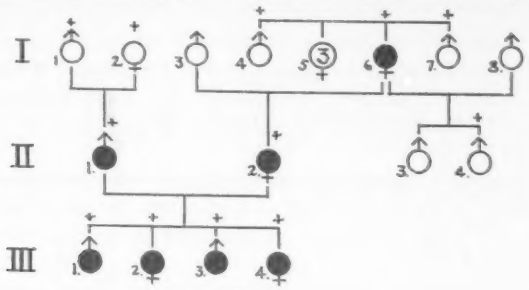


Fig. 17.

aged 2. The father, II, 1, has unilateral epicanthus on left side. The mother, II, 2, illegitimate, has distinct bilateral epicanthus. The paternal grandparents, I, 1 and 2, have no epicanthus. Maternal grandfather, I, 3, is dead. Maternal grandmother, I, 6, has feebly marked bilateral epicanthus; I, 5, three females, are dead. I, 4 and 7 and II, 4 have no epicanthus. All those affected in this pedigree have epicanthus palpebralis.

Fig. 18 shows a pedigree with epicanthus in three generations and direct transmission of the affection.

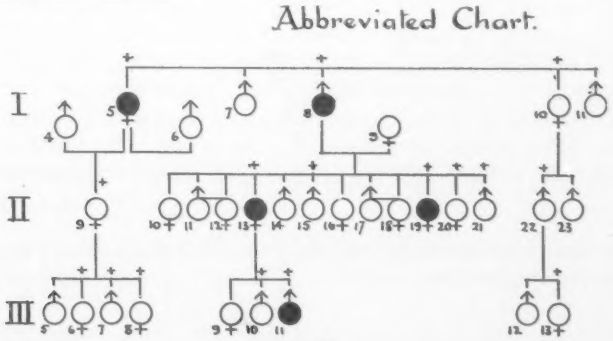


Fig. 18.

Fig. 19. Epicanthus in three generations. Of 11 members who were examined in a sibship of 11, ten have epicanthus. In three of these, II, 1, 8 and 10, the

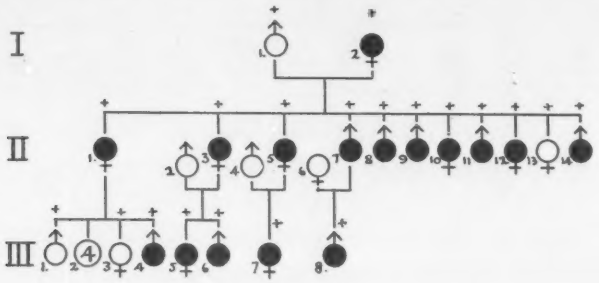


Fig. 19.

epicanthus is unilateral, it has an attenuated appearance and disappears at lower edge of the internal tarsal ligament. Their mother, I, 2, has bilateral epicanthus. III, 4 has left unilateral epicanthus; III, 5 and 6 bilateral epicanthus palpebralis, and III, 7 and 8 bilateral epicanthus tarsalis.

An example of what appears to be discontinuous inheritance is seen in Fig. 20. I, 1, now deceased, is said to have had no epicanthus, but was blind from inflamed eyes in infancy. II, 4 when examined had no epicanthus, which may however have been present in childhood. Other examples are seen in Figs. 14 and 16 on pp. 12, 13. It is important to recollect that if certain parents in these pedigrees had been examined in childhood, epicanthus might have been found, in which case the inheritance would be continuous.

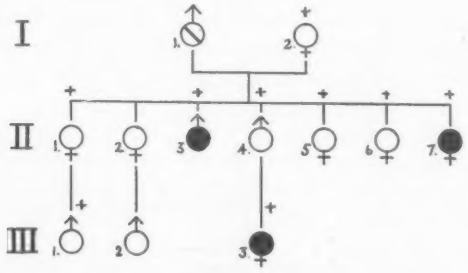


Fig. 20.

The next three pedigrees show the abnormality limited to members of a single sibship.

Fig. 21 shows two normal parents with six children, of whom two males and two females have epicanthus.

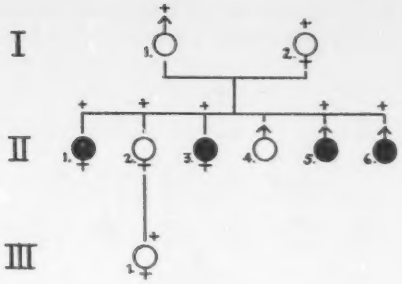


Fig. 21.

Fig. 22. A normal woman, I, 2, had a normal child by her first husband and nine children by her second husband, of whom seven have epicanthus.

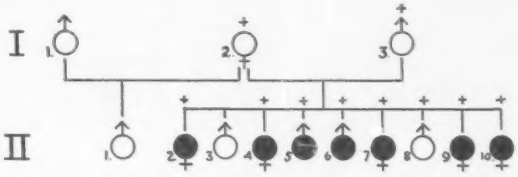


Fig. 22.

Fig. 23. Four affected sons of normal parents. A fifth son, II, 1, the oldest, be it noted, has no epicanthus. Other five sibs, II, 2, 3, 6, 7, 10, died in infancy.

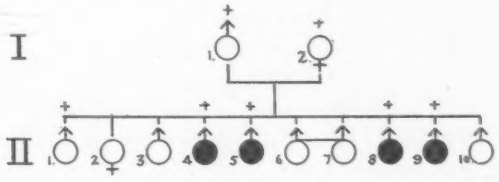


Fig. 23.

Unilateral epicanthus may result from the natural disappearance of an epicanthus from the opposite side. The artificial disappearance of an epicanthus may be the result of operation. If the epicanthus be removed at an early age from both sides it is conceivable that in adult life such an individual would be unaware of his previous condition, which might never have been reported to the investigator of the pedigrees.

Examples of persistence of an epicanthus for a considerable number of years are seen in the next four pedigrees, two of which do not show heredity.

Fig. 24. Dorothy B. T., II, 4, at age of 3 had a bilateral epicanthus extending from tarsal fold and curving downwards to inner end of lower lid, which was still present at age of 28.

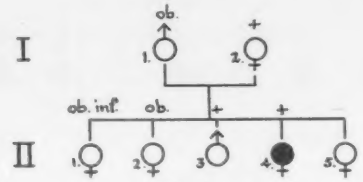


Fig. 24.

Fig. 25. Margaret S., II, 7, a twin, at age of 9 had a unilateral epicanthus which was still present when she was 19 years of age.

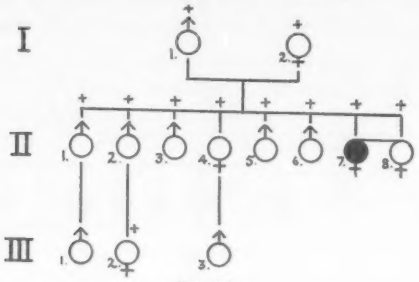


Fig. 25.

Fig. 26. II, 4, a male, the first born of triplets, examined at 5 and 28 years of age. Bilateral epicanthus was present on both occasions.

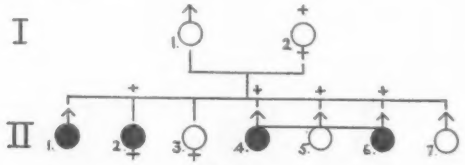


Fig. 26.

In pedigree, Fig. 27, five of six sisters with epicanthus were re-examined 24 years later. In four of them epicanthus was still present. The oldest of these was 33.

The following results have been obtained from the 22 new pedigrees, of which a number of charts have been shown. Consanguinity occurs in one pedigree. The number of individuals with epicanthus is 111, of whom 53 are males and 58 females. Transmission from parent to child occurs in 13 pedigrees, by the father in 6 sibships and by the mother in 14. Discontinuous inheritance occurs in 8 pedigrees, including 5 of those that also show continuous inheritance. Epicanthus is confined to members of a single sibship in 6 pedigrees. Though epicanthus is transmitted

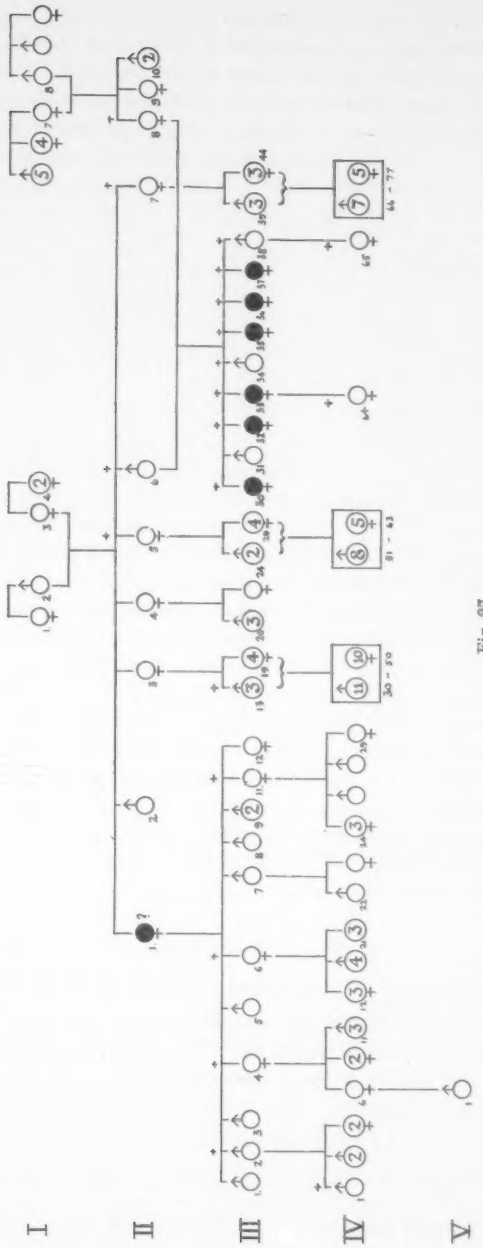


Fig. 27.

from an affected father, or mother, to the children in only a small majority of the pedigrees, transmission is in accordance with what is seen so markedly in Meirowsky's large pedigree. Furthermore, as epicanthus is known to disappear as age advances in a considerable number of people—see Martin's table—there can be little doubt that some of the remaining 9 pedigrees would have shown direct inheritance had all the examinations been made at an early age. When these results are added to those from published pedigrees, the following figures are obtained: Number of affected persons 180; of these 88 are males, 81 females, and in 11 the sex is unrecorded. There are 32 pedigrees in all*, in 21 of these inheritance is continuous, with transmission through males 19 times, and through females 24 times. Discontinuous inheritance alone occurs in 3 pedigrees, and both continuous and discontinuous inheritance in 7. When it is considered that 21 of the 32 pedigrees show transmission from parent to child and that a proportion of normal parents of affected sibships may in childhood have had the abnormality, there is good reason to believe that usually in inherited epicanthus the transmission is continuous.

Changes must occur in the appearance of an epicanthus, which, though well marked at first, finally disappears. No doubt ophthalmologists see the different stages of retrogression, but are the appearances presented recognised as those of an epicanthus which will later disappear? The ill-defined skin folds which in some instances are difficult to diagnose as epicanthus are possibly late stages of a formerly well-defined epicanthus. Cases have been recorded in this memoir of well-marked epicanthus, which after some years appeared to be represented by merely short thin skin folds at the site of the upper part of the vanished epicanthus. Forster† had doubt as to whether an obliquely lying skin fold which began at the internal tarsal ligament and extended upwards and outwards above the tarsal fold was an epicanthus, though he regarded the fold as probably an epicanthus. He saw this oblique fold in five cases, aged respectively 23 (2), 20, 37, 19 years. It may be suggested that these were possibly late stages in the disappearance of an epicanthus palpebralis. The age of the cases supports such a view. It is obvious that a thorough investigation of the inheritance of epicanthus in a pedigree requires not only an examination of each individual, but that this should be made at an early age.

Simple Epicanthus.

A short Account of 22 unpublished Pedigrees which show Heredity.

Fig. 12 (p. 11), Forbes family. Epicanthus in three sibships of two generations. Wm., III, 20, age 15, Isobel, III, 23, and Catherine, III, 24 have left unilateral epicanthus; Ann, III, 26, has bilateral epicanthus more marked on left side. Their mother, II, 5, has no epicanthus. One of the twins, III, 21 and 22, died at birth, the other and Patricia, III, 25, and Alice, III, 27, have no epicanthus. Douglas Y.,

* Figs. 25 and 26 are not included in this statistical summary as the author considers that they do not show heredity.

† Forster, A., *Anatomischer Anzeiger*, Bd. 52, No. 3/4, S. 49—63, 1919.

III, 28, age 4, and his brother William, III, 29, age 11 weeks, have bilateral epicanthus. Their mother, II, 8, has no epicanthus (1934), yet when seen (1913) at the age of 5, epicanthus on left side was recorded, right side was normal. I, 1, father of II, 8, and II, 7, deceased. Besides those marked in the chart, 20 other members were examined, including four females the oldest in sibship II, 1—8 and their children, and also I, 2, the mother of II, 8. None of these had epicanthus. No consanguinity.

Fig. 21 (p. 15), Gerrie family. Four with epicanthus in a sibship of six. Mary G., II, 1, in 1914, at age of 5, had marked bilateral epicanthus which was absent in 1934. Mrs L., II, 2, and her child, III, 1, have no epicanthus. Alice G., II, 3, age 21 years, has epicanthus on right side, none on left, extending from tarsal fold above to just below inner end of lower lid. Richard G., II, 5, age 15, with bilateral epicanthus; distance from one epicanthus edge to that of the other is 38 mm. and from outer canthus to edge of epicanthus is 25 mm. James G., II, 6, age 13, has bilateral epicanthus; distance from edge of one epicanthus to that of the other is 38 mm. and from outer canthus to edge of epicanthus is 23 mm. I, 1 has no epicanthus. It is not stated whether I, 2, now dead, who came with a daughter in 1914, was examined for epicanthus, though this is likely. No consanguinity.

Fig. 28, Mack family. III, 4, male, age 3 (1905) was brought for a definite but not very marked bilateral epicanthus. In 1911, R. and L. fundus normal,

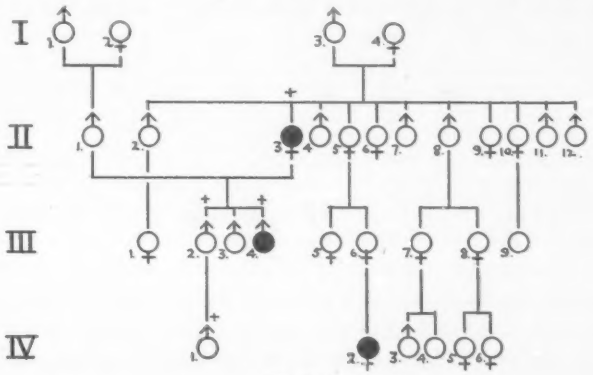


Fig. 28.

refraction H. in each R.V. with $\frac{2{\cdot}50 \text{ D. sph.}}{1{\cdot}50 \text{ D. cyl.}} = \frac{6}{12}$, L.V., with $\frac{2{\cdot}50 \text{ D. sph.}}{1 \text{ D. cyl.}} = \frac{6}{18}$; is mentally deficient. His mother, II, 3, now deceased, had an epicanthus on right side only. She stated that it was a family complaint and that her sister's children, III, 5, and 6, have it. When the mother, II, 5, of these children was written to in 1934 she replied that it does not exist in herself or her daughters, nor in any of her brothers and sisters, nor in any member of her brother L.'s, II, 8, family, but in her granddaughter, IV, 2, 16 months old, she could see a faint indication of it.

"I don't suppose anyone else would notice anything unusual in her left eye, but I have a practised eye so to speak, having been used to seeing it in my sister and her son. I don't understand Mrs Mack, II, 3, saying the same affection was apparent in her sister's children. She must have been mistaken." I, 3 and I, 4 had no epicanthus and none of the photographs or paintings of the earlier generations indicates the presence of an epicanthus. No consanguinity.

Fig. 16 (p. 13), Evans family. Pedigree showing both continuous and discontinuous descent. I, 2, whose photograph shows epicanthus, married twice and had offspring with epicanthus from both marriages. He and his first wife are dead. His second wife, I, 3, has no epicanthus. The following have bilateral epicanthus: Robert E., II, 8, age 10, form palpebralis, has internal concomitant strabismus; III, 2, female, has a minor degree; II, 2, and her son, III, 3, form tarsalis; III, 6, male, age 4, and III, 7, male, age 2 years, form tarsalis; III, 4 and 5, females, ages 8 and 7, in a minor degree; III, 8, female, age 4, form palpebralis. II, 6, male, age 22; II, 7, female, age 14; III, 9, female, age 2; II, 1, 3, and 4 have no epicanthus. No consanguinity.

Fig. 29, McGuffie family. Three brothers, III, 1, 2, 3, in a sibship of three with bilateral epicanthus, and a paternal uncle and possibly their father with the

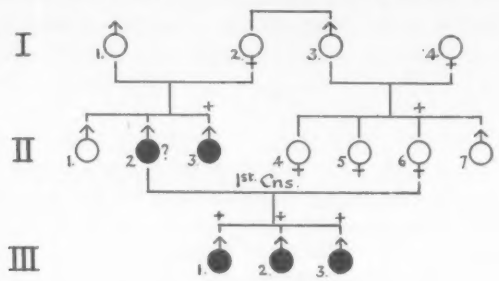


Fig. 29.

same condition. The brothers' ages are from 10 to 2 years. Their father, II, 2, deceased, his wife says had folds of skin similar to those present in her boys. She, II, 6, has no epicanthus. The uncle, II, 3, has attenuated bilateral epicanthus. The boys' parents were first cousins, their paternal grandmother, I, 2, and maternal grandfather, I, 3, being sister and brother. I, 1, 2, 4 and II, 1 are deceased. Consanguinity.

Fig. 17 (p. 13), Stewart pedigree. Epicanthus in three generations; disappearance of epicanthus in one of the members. III, 1, Albert S., age 7, has remains of an epicanthus on left side only. In infancy and also at age of 5 years when seen by Professor Low he had bilateral epicanthus. His brother, III, 3, and sisters, III, 2 and 4, have bilateral epicanthus best marked in the younger sister, age 2 years. Father, II, 1, has left-sided unilateral epicanthus. Mother, II, 2, illegitimate, has distinct bilateral epicanthus. Paternal grandparents I, 1 and 2, normal.

Maternal grandfather, I, 3, deceased. Maternal grandmother, I, 6, has feebly marked bilateral epicanthus. I, 5 are deceased. I, 4 and 7 and II, 4 are normal. Epicanthus palpebralis was the form in each affected person. No consanguinity.

Fig. 11 (p. 10), Findlay pedigree. Epicanthus in the three youngest of six sisters. II, 5, Daisy F., bilateral epicanthus seen at 6 months and 3½ years of age. II, 4, Jessie F., and II, 6, Rose F., age 1 year, have bilateral epicanthus. II, 1 died in infancy; Marjory F., II, 2, age 12, and Dorothy F., II, 3, are normal. Father, I, 1, and mother, 1, 2, have no epicanthus. The former has congenital coloboma at nasal side of right iris and microcornea of same eye. R.V. with high power + lens = reading small newspaper print at 10". Seen in bed with cardiac disease. No consanguinity.

Fig. 13 (p. 11), Cleveland pedigree. Shows direct transmission from mother to children. The family lives in Aberdeen. Sheila C., III, 1, firstborn, age 11, and Jill C., III, 2, have no epicanthus. Jaqueline C., III, 3, age 6, and III, 4, an infant, have marked bilateral epicanthus. Mother, II, 2, has definite bilateral epicanthus. She stated that her mother, I, 1, who lives in England, has the same condition, but this was not confirmed by the family doctor. No consanguinity.

Fig. 14 (p. 12), Brown pedigree. James B., III, 4, age 7, illegitimate, and III, 5, 6, 7 have bilateral epicanthus tarsalis. Mother, II, 4, has no epicanthus. Father, II, 1, is abroad, II, 2 not seen. III, 8, Adam B., age 7, illegitimate, has bilateral epicanthus tarsalis. His mother, II, 5, has it on left side only. II, 6, II, 3, and III, 1, 2, 3 have no epicanthus. III, 1 is aged 6. No consanguinity. I, 1 and I, 2 have no epicanthus. Had II, 5 bilateral epicanthus at an earlier age and did II, 4 formerly have epicanthus?

Fig. 15 (p. 12), Polson pedigree. A boy and two maternal uncles with epicanthus. Parents, III, 2 and 5, and maternal grandmother, II, 3, have no epicanthus. Gordon H., III, 12, age 16, has a minor degree of epicanthus bilateralis. Neil H., III, 14, also has bilateral epicanthus better defined on right side. Lizzie H., III, 8, has broad root of nose, but no fold at either inner canthus. Paternal grandmother, II, 1, very broad from one inner canthus to the other, no epicanthus, skin wrinkled. IV, 4 and 5, no epicanthus. IV, 6, III, 6, and IV, 1 died in infancy. III, 9 married, no offspring.

Fig. 30, Calder pedigree. Showing both direct and indirect inheritance. Nan C.,

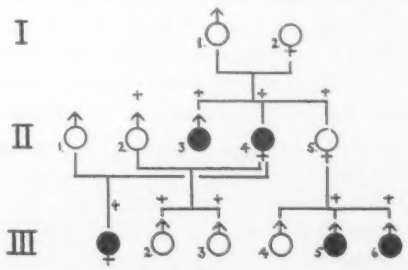


Fig. 30.

III, 1, age 9, illegitimate; her mother, II, 4, and uncle, II, 3, unmarried, have bilateral epicanthus tarsalis. II, 4, not consanguineous with II, 1, or her husband, II, 2, by whom she has two normal sons, III, 2, 3. II, 5, with no epicanthus, has three sons, one (III, 4) of whom was killed. The other two have epicanthus tarsalis, bilateral in III, 5, age 7, and unilateral on right side in III, 6, age 3. I, 1 and 2, deceased. It is of course open to doubt whether II, 5, at an earlier age, had epicanthus. No consanguinity.

Fig. 20 (p. 14), Jamieson pedigree. Epicanthus in a child, her paternal uncle and a paternal aunt. Disappearance of an epicanthus from operation. Jas. J., II, 3, age 28, unmarried, at age of 3 had marked bilateral epicanthus which covered the caruncles; distance from edge of one epicanthus to that of the other was 26 mm. Excellent result after Wicherkiewicz operation on each. Annie J., II, 7, age 12, with unilateral left-sided epicanthus, has internal concomitant strabismus. Irene J., III, 3, age 2, with unilateral left-sided epicanthus. Father, II, 4, has no epicanthus. I, 1, deceased, his wife states had no skin fold, worked at blind asylum all his life, eyes inflamed in infancy. No consanguinity.

Fig. 26 (p. 16), Nicol pedigree. Epicanthus in four members of a sibship of seven. II, 4, 5, 6 are triplets, age 28; of these, the first born, II, 4, and the last born, II, 6, have bilateral epicanthus, which was first seen in II, 4 when he was examined at 5 years of age. He was brought for divergent squint which passed off; refraction of each eye was +1 D. Their mother says that the two triplets, II, 4 and II, 6, were remarkably alike. II, 5 has no trace of epicanthus. II, 2, female, age 30, has marked epicanthus on left side; on right side is merely a trace. In all these cases the epicanthus arises from the upper lid tarsal fold. II, 1, male, age 31, not seen, but, to judge by the account of his brother, II, 4, and a photograph, he has bilateral epicanthus best marked on left side. Father, I, 1, deceased. Mother, I, 2, has no epicanthus. II, 3 died in infancy; II, 7 is abroad, age 26; photograph shows no epicanthus. No consanguinity.

Fig. 22 (p. 15), Maitland pedigree. Normal parents with seven children in a sibship of nine with epicanthus. Mary M., II, 2, age 13, with bilateral epicanthus. II, 3 died in infancy. II, 4, age 9, has feebly marked bilateral epicanthus, nose well developed. Father says the skin folds were formerly more marked in her than in any of the others. John M., II, 5, Edward M., II, 6, and Daisy M., II, 7, have bilateral epicanthus. James M., II, 8, is normal. Margaret M., II, 9, and Frances M., II, 10, age 7 months, have bilateral epicanthus. These are Mrs M.'s, I, 2, children by her second marriage with M., I, 3. By her first marriage with W., I, 1, she had a son, T. W., II, 1, an only child without epicanthus. Neither he nor his parents have epicanthus. Both I, 2 and I, 3 are normal. No consanguinity.

Fig. 18 (p. 13). Lemmon pedigree. Showing continuous transmission in three generations. I, 8 and his daughter, II, 13, have bilateral epicanthus, and his grandson, III, 11, right-sided unilateral epicanthus. II, 19, when brought for internal concomitant strabismus when 5 years of age (1916), had well-marked bilateral epicanthus, which is apparently unchanged (1934). It covers the plica

semilunaris and caruncle, arises from tarsal fold of skin on upper lid, and is attached below to the skin down and in from inner end of lower lid. I, 5, Mrs G., with bilateral epicanthus, had an illegitimate daughter, II, 9, by a half-caste, I, 4; she married I, 6, but had no more children. II, 9 and her children, III, 6, 7, 8, have no epicanthus. II, 21, the youngest in the sibship, is 19 years of age. Abroad: I, 7; II, 2, 4, 10. Married without issue: I, 6; II, 5, 7, 10. Deceased: I, 1, 2, 11; II, 6, 11, 12, 14, 16, 17, 18; III, 5, 9. No consanguinity. Some of those referred to are not shown in the abbreviated chart.

Fig. 31, Dunbar pedigree. III, 4, Susan D., age 14 months, with well-marked bilateral epicanthus and alternating internal strabismus, is the child of a man,

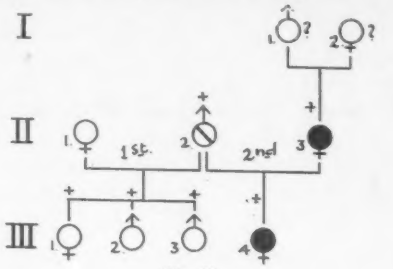


Fig. 31.

II, 2, age 49, with congenital club foot, and a woman, II, 3, with bilateral epicanthus. By his first wife, II, 1, he, II, 2, had three normal children, III, 1, 2, 3, ages 16, 14, 10 years. I, 1 and 2 are dead; no information regarding the presence of epicanthus in them is obtainable. No consanguinity.

Fig. 23 (p. 15), Stewart family. Epicanthus in four brothers in a sibship of ten. II, 9, George S., age 6 (1932), with marked bilateral epicanthus and right convergent internal strabismus. When seen again two years later, it was noted that the epicanthus covered the plica semilunaris and caruncle. It extended from tarsal fold of upper lid to below inner end of lower lid. II, 8, Gordon S., age 11, bilateral epicanthus more marked on left side. II, 5, Robert S., age 14, bilateral epicanthus better marked on left side. II, 4, Wm. S., poorly marked bilateral epicanthus. Died in infancy: II, 2, 3, 6, 7, 10. Parents, I, 1 and 2, normal. No consanguinity.

Fig. 32, Massie pedigree. A mother and two sons with bilateral epicanthus.

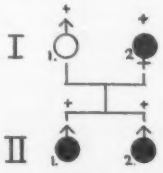


Fig. 32.

II, 1, Alexander M., age 10, and his brother Jim, II, 2, age 8, to a less extent, have bilateral epicanthus. The tarsal fold is continued downwards to level of lower

lid margin, where it ends on side of nose. Their mother, I, 2, presents the same appearances, caruncle and plica normally developed. Father, I, 1, has no epicanthus. No consanguinity.

Fig. 33. Rey pedigree. Bilateral congenital epicanthus in a female, her niece, and paternal grandmother. III, 9, A. R., female, age 22, has well-marked bilateral

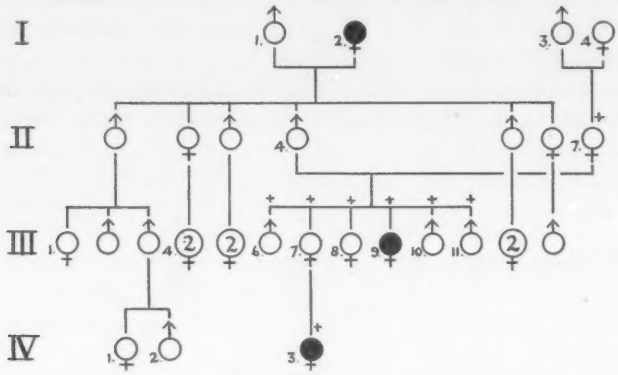


Fig. 33.

epicanthus continuous with tarsal fold of upper lid, no ptosis, caruncle visible, lower punctum not displaced outwards. None of her three brothers, III, 6, 10, 11, or two sisters, III, 7, 8, shows the anomaly, but her niece, IV, 3, has definite bilateral epicanthus continuous with tarsal fold of upper lid. I, 2, her paternal grandmother (not seen), now dead, had exactly the same condition. When A. R., III, 9, was born, the then family doctor (deceased) reminded her mother, II, 7, of the condition in I, 2. II, 7 says that her mother-in-law was able to stretch the epicanthus by raising her brows. She, II, 7, and her husband (deceased) had no epicanthus. She was fifth born in a sibship of eight; neither of her parents nor any progeny of her brothers and sisters (not shown in chart) had epicanthus. III, 1 and the elder of the two sisters, III, 4, are married, without issue. No consanguinity.

Fig. 34, Donald pedigree. Father and two sons with bilateral epicanthus. II, 1, age $3\frac{1}{2}$, has also left internal concomitant strabismus. In the father, the epicanthus

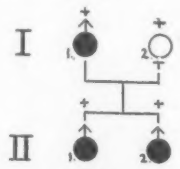


Fig. 34.

begins below the tarsal fold of upper lid and curves downwards to a point below the inner canthus. It has a more solid appearance than usual.

Fig. 27 (p. 17), Gray pedigree. Six sisters with epicanthus. Re-examination of five of them after an interval of 24 years. III, 30, Mary G., age 15 (1912), had a minor degree of epicanthus more marked on left side, which had disappeared by 1934; bridge of nose well developed. III, 32, Joan G., age 9 (1910), brought for blepharo-conjunctivitis, had well-marked bilateral epicanthus which had not altered by 1934. III, 33, Jeannie G., age 8 (1912), with bilateral epicanthus which had not altered by 1934. Her daughter has no epicanthus. III, 35, Daisy G., age 6 (1912), with bilateral epicanthus, which by 1934 had disappeared from left side and was fully marked on right side. III, 36, Annie G., age 4 (1912), with bilateral epicanthus which was still present in 1934. III, 37, Williamina G., age 2 (1912), with unilateral left-sided epicanthus, in 1934 was abroad. III, 31, male, died age 2 (not seen), had no epicanthus. III, 34 and 38 have no epicanthus. None of those affected in the sibship is married except III, 33. Both parents, II, 6 and II, 8, are normal. The mother, II, 8, now dead, said in 1912 that her husband's sister, II, 1, deceased, and one of her daughters had the same condition as her own daughters. This could not be confirmed in 1934 as the three daughters, III, 4, 6, 11, had then no epicanthus. A fourth daughter, III, 12, died young. I, 1, female, age 94 (1912), knew nothing of the condition in any of her relatives when interviewed by II, 8. No consanguinity.

Fig. 19 (p. 14), McLean pedigree. Sixteen affected members in three generations. Transmission continuous through male or female. In a sibship of 11, epicanthus bilateralis was present in seven; in one there was no epicanthus, and in three (II, 1, 8, 10) only feebly marked unilateral epicanthus was found. I, 1, age 56, is normal; I, 2 has bilateral epicanthus palpebralis which has a flattened appearance, skin is thin. III, 2, one died in infancy, three still-born. III, 1, age 10. III, 3, age 3. III, 4, age 10 months, has left unilateral epicanthus. III, 5, age 3 years, and III, 6, age 4 months, have bilateral epicanthus palpebralis. III, 7 and III, 8 have bilateral epicanthus tarsalis, and so have II, 5 and II, 7, but not so markedly. No consanguinity.

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GROWTH CHANGES IN BODILY SIZE AND PROPORTIONS DURING THE FIRST THREE YEARS: A DEVELOPMENTAL STUDY OF SIXTY-ONE CHILDREN BY REPEATED MEASUREMENTS.

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I. INTRODUCTION.

Investigators of the problems of growth changes in bodily size and proportion have frequently pointed out that seriatim measures, in a group large enough for statistical treatment, are greatly to be desired. Such measures over the entire growth period would, of course, require many years; in consequence, most seriatim studies cover only a few years of the growth cycle, or are limited to few cases. These studies have usually been limited further by restriction to measurements of height and weight only.

The present study presents cumulative data for the first three years of life. During these early years growth is, of course, much more rapid than at any later age, and consequently a greater portion of it is included for a given unit of time.

The use of a constant sample through a three-year period presents many advantages. With a constant sample we need not make allowances for sampling fluctuations from one age level to the next, and so may work with a smaller number of cases than would otherwise be necessary for securing growth and variability trends. From repeated measures, we can determine the stability of growth rates in individuals. We can study the relationship between growth rates of different parts of the body, for the group as a whole, and for individual members in the group. We can study the consistency in types of build between the earlier and later ages measured. Finally, as a result of these, we may learn, in part, the extent to which one may predict adult size and body build in infancy.

If, as a number of students of body build have claimed, build and certain mental characteristics are related, findings of inconstant body characteristics would suggest a search for concomitant variations in personality traits.

Anthropometric studies of infants are attended with certain serious difficulties. One of these is the problem of securing normal unselected samples. This difficulty is decreasing in recent years, with the prevalence of Well-baby clinics, and the direction of attention to the importance of infant health and welfare. Another difficulty lies in the comparative unreliability of anthropometric measures of infants. Infant measures are less reliable than those of adults because they are smaller; the

importance of chance variations is thus comparatively greater, even though the absolute value of these variations may be the same. But an even more important factor is that of the emotional reactions of the infants to the procedures of measuring. Before they reach an age where they can understand what is wanted of them and are no longer afraid of the measuring instruments, many infants protest, more or less vehemently, against being held in position for the measures and against the application of the instruments. Besides this is the fact that the examiner is usually not well known to the child, and that children between the ages of six months and two years or so of age usually dislike being handled by strangers. The result is that the measures must at times be made on a tense, struggling, crying child. This situation adds greatly to the difficulty of making accurate, reliable measures. Often the measure must be caught in a fleeting moment when the child is in the correct position, and comparatively quiescent. As Baldwin (5) has pointed out, however, much can be done by the examiner, in the manner of his approach, to put the child at his ease and so secure more accurate measures*.

Since normal samples of healthy infants are now available, and the examiner may learn a method of approach which diminishes the child's resistance to being measured, the only remaining question is that of the reliability of the measures, as such. Bakwin and Bakwin (1), who recently made a large number of measures on infants under one year, repeated a series of measurements on the same children and found what they felt was a high degree of reliability—at least as reliable as Todd (38) and others have found for adults. For ten repeated measurements of from 11 to 20 infants on total body length, sitting height, diameter of face, bicristal diameter, biacromial diameter and circumference of thorax at nipples, the coefficients of variation ranged from ·04 per cent. for length to 1·3 per cent. for bicristal diameter†. It seems possible, then, that the disturbing factors can be overcome, and that a study based on the anthropometric measurements of infants may be of some significance.

We are quite familiar, now, with the difference between infants and adults in body proportion, and some of the changes they undergo in the process of maturation. The relatively greater maturity of the upper end at birth, and the cephalo-caudal trend of development is true, in general, of body proportions (Godin (19), Bardeen (7), Bean (8)), as it is of neural and motor development (Coghill, Shirley). However, there is still much to learn about the relative growth rates of the different parts, especially in the same individuals. Seriatim measures have not hitherto been made on a sufficiently large number of infants to allow adequate statistical treatment. Valuable studies of proportional changes with growth (based, for the most part, on samples which are made up of different individuals at successive ages) have been made by Scammon and Calkins (35) for foetal growth, Scammon and others (24) for

* This does not necessarily mean carrying on conversations with a child as Gray and Ayres (23) have inferred, but is often a matter of the examiner's attitude, which may vary according to the child.

† Their data have not, however, been treated in such a way as to make possible comparisons between remeasures on the *same* individual, since the measures on all infants have been grouped together.

post-natal growth, and Boas (10) for rates of growth in height and weight. Baldwin (5) (p. 107) has done this for a large number of seriatim measures on a few children. Godin (19) has shown diagrammatically the changing proportions from the foetus to the adult; the foetus has a proportionately large head and short extremities, especially legs, and the relative proportions of these change strikingly with growth.

Most of the above-mentioned studies have been concerned with the rates of growth and changes in proportion to the various body parts. A closely related field is that of the consistency in growth rates within the individual. Baldwin (5) has found, for example, that children who are tall for their age continue to be tall for their age as they grow older, while short children remain relatively short. Boas (10), on the other hand, has found recently that persons of the same adult height reach that height by different growth rates, and with different ages of most rapid growth. Hence, "acceleration or retardation within the same social group has apparently no influence upon adult stature." Merrell (31), studying rabbits, also found no relationship between rapid early growth and adult size. Studies of consistency must, necessarily, be made from seriatim measures, and must concern themselves with the independent growth curves of individuals.

Another interesting field for investigation is that of body build. Only a few studies of build have been made on infants. Bakwin and Bakwin (4) claim to have found a twofold division of body types in infants. Lederer (29) has made an elaborate classification of four types, giving them names which imply characteristic accompanying behaviour patterns or mental traits. There is considerable disagreement among investigators of body build in adults, as to whether there are distinct types, and, if so, what the types are. We may expect to find the same disagreement in respect to infants, but if we can choose a criterion (or criteria) for measuring individual differences in infants' build we may then determine whether those differences remain constant, and whether they are related to other mental or physical characteristics.

The height and weight of infants under one year have often been measured; and norms based on a large number of measures have been presented. But additional anthropometric measures on such young subjects are few, though there have been several recent studies of infants in which a large number of measures were made. Probably the most complete series of measures has been reported by the Iowa Child Welfare Research Station (26). Recent studies by Freeman (18a, 18b) and Freeman and Platt (18c) give means, S.D.'s and indices for a large number of measures made at frequent age levels on children from birth to 14 years. Other studies are those of Bakwin and Bakwin (1, 2, 3), of Boyd at Minnesota, and of Thompson at the Yale Psycho-Clinic. The results of these last two studies have not yet been published. Other studies, not quite so recent, are those of Fleischner (13), Montague and Hollingworth (32), and Taylor (36).

II. THE PRESENT STUDY.

A. SAMPLE.

The present study was undertaken with a group of sixty-one infants who were measured at monthly intervals from one to twelve months, and at fifteen, eighteen, twenty-four, thirty and thirty-six months of age. They were measured on, or within a few days of, their birth date. During the first year the observations were made, if not on the birth date, within four days of that date. Longer deviations from the birth date were allowed later, if necessary (in the third year as much as two weeks), but these wide deviations were rare. This method of seriatim measures made possible a comparison of growth tendencies at different stages of development, based on measures of a sample which was identical, or nearly so, throughout the series. There were occasional absences, and a number of the children moved out of town before the end of the three-year period. Forty-nine of them were still enrolled in the study at thirty-six months.

The method of obtaining the sample caused some selection, making for a somewhat homogeneous group, although originally an attempt was made to get babies from families representative of a cross-section of the community. The subjects observed form a large proportion of Berkeley babies born in two hospitals during a six-month period. Of these babies, we included only those whose parents were white, English-speaking, and willing to co-operate by monthly visits to the Institute of Child Welfare for a series of observations. These observations included, in addition to the anthropometric measurements, tests of mental, motor and physiological development.

A comparison of this Growth Study group with a larger group of Berkeley families in which infants were born at about the same time*, show to what extent they are representative of the Berkeley population. This comparison shows that the families of this smaller group average higher in annual income, education, and occupational rating than the unselected population of families into which babies were born during this time in Berkeley. The mean annual income of the Growth Study is $2844, as compared with the Berkeley Survey mean of $2544. The mean educational level of the Growth Study parents is, for the fathers 13·8 years and for the mothers 13·1 years, while the Berkeley Survey fathers average 12·2 years and the mothers average 11·6 years of schooling. Thirty per cent. of the fathers of the Growth Study infants were classified in professional or executive occupations, as compared with 22 per cent. of the Berkeley Survey group. An equal proportion of both groups were classed in "white collar" occupations, while the smaller group had comparatively fewer in the skilled, semi-skilled and unskilled labour categories.

From these comparisons, it may be assumed that this group of infants comes, for the most part, from homes where the education and the income combine to insure hygienic living conditions and sufficient nourishing food; and hence, well-nourished, healthy babies. This selection may account, in part, for the fact that

* *The Berkeley Survey*, a study made by Frances M. Welch and Agnes Covalt.

the mean measures of this group of Berkeley babies are larger, after the sixth month, than those of other published infant norms (41).

The group is probably also, because of similar racial stocks and freedom from environmental handicaps, more homogeneous than would be an unselected sample. For this reason many of the correlational values between measures will be more significant than the same values would be for a more heterogeneous group of infants.

B. The Measures: Instruments and Technique.

The measures which we undertook to make were selected on the basis of two criteria: the accuracy with which they could be made on infants, and their probable value in indicating growth trends in size and bodily proportions. The measures taken were length (reclining during the first eighteen months, afterwards standing); stem length (reclining during the first eighteen months, afterwards sitting); widths of shoulders, chest and hips; chest depth; circumferences of head, chest and abdomen; head diameters, transverse and antero-posterior; and weight. This selection of measures and the methods of taking them are based, for the most part, on those used at the Iowa Child Welfare Research Station. As the technique has been described in a recent publication (26), we will indicate here only our deviations from it. In the present study we have not considered the abdominal circumference or the head diameters.

The instruments used during the first eighteen months varied from those used at Iowa in several respects. Chest depth was measured with the spreading callipers; and the shoulder, chest and hip widths with the small sliding callipers. This change was made because the large sliding callipers were found to be awkward with the young infants. From eighteen months on, the large callipers were used in the chest, hip and shoulder diameters. Circumferences at all ages were measured with a linen tape graduated in millimetres, and having a spring attachment at one end by which the pressure could be kept constant*. The tapes were replaced at intervals to prevent inaccurate measures due to their shrinkage with use. Standing length and sitting stem length were measured on a stadiometer made by the Continental Scale Works in Chicago. Since this stadiometer was calibrated in inches, the measures were read in inches and quarter-inches, and converted to centimetres for treatment of the data. Infants under about twenty pounds were weighed on a light portable spring scale whose accuracy was checked at frequent intervals; the heavier babies were weighed on a balance scale. Both scales were calibrated in pounds and ounces. These have been converted to kilograms for treatment of the data. All measures were made with all clothing removed. At about nine months, it was found to be impossible to measure most children while lying down, without vehement protest. For this reason, the head, chest, and shoulder measures were made with the child sitting, until he was able to stand with support, when all of the body diameters and circumferences were taken standing. After eighteen months, the lengths also were taken standing.

* These are made by the Narragansett Machine Co., Providence, R.I.

Further differences in procedure on individual measures are as follows: For length* of the stem in the erect posture the child sat in the stadiometer, with its spine in contact with the stadiometer, and knees flexed by placing its feet up on the platform in the position described by Dreyer (17). The experimenter then held the head in place and measured in the same way as for standing length. *Chest depth:* when the spreading callipers were used, the fixed branch was placed on the sternum at the nipple line, and the movable branch at the centre back, just below the scapulae.

In order to obtain accurate measures, every effort was made to keep the child quiet and contented. An active restless child would not maintain the optimum position long enough for a measure to be read, and the chest measures of a violently crying child varied far more widely than those of one breathing quietly. The difficulty of obtaining accurate measures increased as the infants grew older and more active until, at eighteen or twenty-four months, they could be made interested in the procedure and persuaded to stand still and submit willingly to being measured. The whole situation was made more difficult by the fact that mental, physical and motor tests and the taking of photographs were a part of the procedure during the period of observation, so that the child was often excited or tired before any anthropometric measures were taken. The experimenter measured the child with as little handling as possible, and always with a gentle friendly approach. But from about seven to eighteen months, the situation was complicated by the child's emotional reaction—at times violent—to the strange situation and to being handled by strange persons. In the difficult cases the mother usually co-operated by holding the child in position. These variable factors probably resulted in some inaccuracies in measures. For this reason, measures taken under unfavourable conditions, when they deviated widely from other measures of the same child, were not used in the treatment of the data.

Every effort was made to overcome unreliability of the measures, by carefully controlling the position of the child, and by securing its acquiescence whenever possible. Some of the measures, especially weight and height, could be made fairly accurately, even though the children were resistant, while others were greatly affected by variations in attitude toward the instruments and the examiner. An example of the latter is shoulder width. The child who shrinks from the callipers and draws its shoulders together may decrease its shoulder width by several centimetres. On the other hand, the unreliability of a measure may be a function of the dimension itself. The amount of fat or the firmness of the muscles over the points measured, in hip width, for example, may vary enough in the same child, from one month to the next, to change his relative position in the group. Other measures, such as chest depth, can be changed by assuming different body postures which change the relationships of the component parts involved in the measure. Since children so young cannot be expected to co-operate, or even to understand

* The term "length" has been substituted for "height," in this paper, to apply to both reclining length and standing height; and the symbol L. is used instead of H. in the indices and tables, i.e. $W./L.^3$.

the positions desired by the experimenter, we may expect to find such measures relatively unreliable. The most reliable measures should be those taken from bony points without intervening joints, and with little flesh covering. Some of these may be read accurately, even on a protesting child.

Most of the measures taken in this study were made by one person (Bayley), but in a few instances they were made by the other experimenter in the Growth Study, Dr L. V. Wolff. After determining a given measure, the experimenter read it aloud to a recorder who entered it on the record form.

C. Ratings of Body Build from Photographs.

In order to evaluate the validity of the different body build indices, a criterion was set up on the basis of photographs taken under standard conditions. The use of photographs made it possible to compare all of the children of a given age at one time.

(*a*) *Method of taking Photographs.*

The infants were photographed, with all clothing removed, at ages 1, 3, 5, 7, 9, 12, 15, 18, 24, 30 and 36 months. The pictures were taken with a Graflex camera, indoors, with standard lighting conditions, lens at *F* 4·5, and a ·1 second exposure. The infants at first lay or sat on a canvas table with the camera placed at a standard distance. When the children were able to stand alone, they were placed on "foot-prints" painted on the floor, with the camera at a new standard distance. Three exposures were always made. In the early months two exposures of the child in the dorsal and one in the prone position were taken. When a child was able to sit alone, one exposure in the sitting position was substituted for one of the dorsals. When he was able to stand alone, one front, one rear, and one side view, all standing, were taken. However, it was not always possible to persuade a child to stay where he was put for a picture, so that many of the resulting photographs are not exactly as desired. Usually at least one of the three was sufficiently in position to make possible a fair estimate of body build, while for the 30 and 36-month pictures, better co-operation of the children resulted in very satisfactory pictures.

(*b*) *Method of Rating.*

For the purpose of rating body build from the photographs, the pictures were sorted into groups according to age, with each child's pictures for one age fastened to a single large card. The ratings were made by comparing each child with the others of the same age. To do this the rater sorted the cards into seven stacks, if that many divisions could be made, putting pictures of similar body build together. Each age group was rated separately without reference to other ages, and each characteristic considered was rated independently. Standing pictures were not compared with sitting, nor sitting with reclining pictures.

A rating scale of seven different body proportions was prepared, with a seven-point rating for each. Since only one of these proportions—general chubbiness, or distribution on a lateral-linear scale—could be rated adequately from the

reclining and sitting pictures, it is the only one considered in this paper. Chubbiness was rated for all months at which pictures were taken, independently by two judges, one (*B*) who had measured the infants, and one (*D*) who was not familiar with the children and so was not influenced, in his ratings of the photographs, by any preformed opinions of a child's build. The pictures at 15, 18 and 24 months were rated by the writers and by Dr L. V. Wolff. The scores used for comparison with the objective measures are the combined ratings of the two (or, where available, three) judges.

III. RESULTS.

A. THE RELIABILITY OF THE MEASUREMENTS.

The best measure of the reliability of data such as these would be that afforded by the coefficients of correlation from repeated measurements taken during one examination period. This procedure was not followed in the present instance, for several reasons. In the first place, the amount of time allotted for the examination of a given child, at any one examination period, was too brief to permit the repetition of the several anthropometric measurements and still allow for the other observations which were included in the plan of testing. Again, the attempt to repeat the anthropometric measurements would have led (particularly at certain ages) to emotional disturbances in some of the infant subjects which would have rendered the second series difficult to obtain and often unreliable when they could be obtained.

In the absence of reliability coefficients derived from repeated measurements during a given examination period, the reliability of the measurements reported upon herein may be estimated in the following ways: (1) by a study of the coefficients of correlation obtained when one month's measurements are paired with those taken at the next succeeding examination period; (2) by the remeasurement of a small group of infants after an interval of one week; and (3), less directly, by a consideration of the consistency of the trends of central tendency for the group.

(1) Low coefficients of correlation when successive monthly measurements are correlated with each other could be due either to errors in the taking or in the recording of the measurements, or to pronounced shifts, during the interval between the two examinations, in the relative position of the individuals composing the sample. High correlations, on the other hand, can mean only that the measurements were made with a relatively high degree of precision and that there is a definite tendency toward stability in the rank orders of the subjects from month to month. Table I gives the self-correlations obtained in that way. Weight, total body length, and head circumference are seen to be highly stable measures. Stem length, hip width, and chest circumference rank next. Shoulder width, chest width, and chest depth are relatively unreliable, and this fact needs to be taken into consideration when the respective merits of various proposed indices of body build are under consideration. It may be noted here that our results, based upon seriatim measurements made on living infants, are in agreement with the conclusion of

TABLE I.

Self-Correlations between Consecutive Measurings.

	Months															Mean r
	1×2	2×3	3×4	4×5	5×6	6×7	7×8	8×9	9×10	10×11	11×12	12×15	15×18	18×24	24×30	30×36	
W.	·909	·945	·931	·947	·961	·943	·954	·961	·961	·941	·965	·957	·931	·931	·935	·961	·946
L.	·918	·926	·898	·889	·942	·950	·939	·936	·914	·928	·932	·948	·921	·916	·947	·974	·930
H.C.	·916	·934	·937	·920	·954	·942	·941	·876	·916	·911	·922	·924	·919	·912	·904	·950	·924
S.L.	·772	·847	·821	·751	·879	·836	·860	·875	·887	·904	·883	·863	·871	·784	·750	·779	·835
H.W.	·857	·810	—	—	·794	—	—	·779	—	—	·839	—	·745	·812	·854	·933	·825
C.C.	·829	·821	—	—	·765	—	—	·881	—	—	·837	—	·802	·823	·839	·884	·819
C.W.	·709	·663	—	—	·669	—	—	·808	—	—	·637	—	·545	·638	·765	·874	·702
S.W.	·766	·628	—	—	·699	—	—	·706	—	—	·829	—	·567	·541	·737	·738	·690
C.D.	·670	·710	—	—	·561	—	—	·601	—	—	·580	—	·394	·677	·731	·884	·659
W./L.^{2}	·718	·741	·808	·857	·800	·726	·871	·874	·866	·832	·882	·874	·830	·732	·740	·911	·816

Scammon and Calkins (35), based upon repeated measurements made on human foetuses, that total body length (crown-heel length) is a more highly reliable measure than stem length (crown-rump length). Boyd (13) has also found, on remeasuring Nursery School children, that standing height is by far the most reliable of a series of length measures.

(2) As an additional check on the reliability of the measures, fifteen infants in a San Francisco baby home, ranging in age from nine days to seven months, were measured, and then remeasured by Bayley after an interval of one week. Unfortunately it was impracticable to measure more cases in this way, and since the number of cases is so small, the results must be considered as only tentative, and are interesting, mainly, because they corroborate the results of comparisons between measures made one month apart. Table II gives the means and S.D.'s, and the differences between the first and second measures. The variation is much greater for some measures than for others. The greatest mean difference is in length, the smallest in hip width. Since length is also the longest measure taken, its greater variation does not necessarily invalidate it. When we convert the mean differences into percentage of the mean of each measure (column 9) we find that the head measures are relatively most consistent, while stem length, total length, hip width and chest circumference all have mean differences of less than 2 per cent. of their mean size. Chest width shows the greatest percentage variation (3·6 per cent.), with chest depth and shoulder width also relatively unreliable.

(3) The curves of the means for all measures are very smooth when one considers that they are composed of between 50 and 60 cases at each age. This is due, in large part, to the constant sample. Gross errors, however, either in the taking of the measurements or in their recording, would presumably have led to a greater fluctuation in the means. Such fluctuation would be especially evident in the means for the sexes separately. Table IV and Figs. 2 and 4 show that these sex differences are small but consistent. If they were due to chance errors

TABLE II.

Repeated Measures of Infants in San Francisco Baby Home.*

(Centimetres.)

1	2	3	4	5	6	7	8	9
	No.	M_1	σ_1	M_2	σ_2	M Diff.	σ Diff.	$\frac{M \text{ Diff.}}{M_1}$
Length	13	53·95	3·0338	54·08	2·9690	·75	·4005	·0139
Stem Length	13	35·56	2·0269	35·42	2·3970	·46	·4568	·0129
Shoulder Width	14	15·43	1·8941	15·36	2·1379	·44	·2660	·0285
Chest Width	15	11·37	1·3766	11·23	1·1396	·41	·3025	·0361
Hip Width	15	11·03	1·5098	11·07	1·6774	·18	·2067	·0163
Chest Depth	15	10·04	·9165	10·06	·8648	·29	·2584	·0289
Head Circumference	15	38·22	3·4906	38·15	3·4699	·21	·1752	·0055
Chest Circumference	15	35·96	3·6519	36·09	3·9633	·65	·5530	·0181
Transverse Head	13	9·97	·6467	9·98	·6182	·05	·0480	·0050
Antero-posterior Head	14	13·04	1·0645	13·04	1·0557	·04	·0534	·0031

* The ages of these children are as follows: 2 weeks, 2 cases; 1 month, 2 cases; 2 months, 4 cases; 3 months, 6 cases; 4 months, 1 case; and 7 months, 1 case.

in measurement, the direction of the difference would fluctuate, with the females sometimes exceeding the males, and *vice versa*.

As for the reliabilities of the judges' ratings of photographs, the correlations between the ratings of judges *B* and *D* range from ·71 to ·81, with a mean of ·76 (Table III). The correlations for months 15, 18, and 24 average ·85 between *B* and *W*, and ·78 between *W* and *D*. The combined ratings of all judges were correlated with themselves at consecutive ages, and these *r*'s (Table III), when the necessarily wide age intervals between pictures are considered, are satisfactory. They range from ·65 to ·93 with a mean of ·81. A coefficient of correlation is, of course, always to be interpreted in the light of the known facts with respect to such factors as (1) the size of the sample, (2) the degree of heterogeneity present in the sample, and (3) the extent to which the sample is selected and the kind of selection involved. For example, a selection of children from the poorer social strata might include a much larger proportion of disease-affected and under-nourished children than would be found in a sample from the more prosperous classes of a population. The latter group could very well be subjected to more standardised regimes and diets, and as a result of homogeneous environment have a smaller variability in measures. It seems probable that the present sample is, in the respects of diet and regime, more homogeneous than the population of the whole community. When we consider this, and the relatively small number of cases, these correlations, based on a seven-point rating scale, show a fair relationship between the ratings of the different judges; and when the judges' ratings are combined into a single score, we have what we may consider a reliable criterion of body build against which to compare the objective indices.

TABLE III.

Correlations between Judges' Ratings of Chubbiness.

	Months											Mean r
	1	3	7	9	10	12	15	18	24	30	36	
$B \times D$	·77	·73	·75	·79	·79	·75	·81	·79	·71	·74	·73	·76
$B \times W$	—	—	—	—	—	—	·88	·88	·78	—	—	·85
$W \times D$	—	—	—	—	—	—	·77	·79	·78	—	—	·78

Self-Correlations of combined Judges' Ratings.

Months								Mean
1 & 3	3 & 5	5 & 7	7 & 9	9 & 12	12 & 15	15 & 18	18 & 24	
·70	·79	·65	·83	·93	·85	·86	·85	·81

B. Growth trends of the Individual Bodily Dimensions.

1. *Central Tendency.*

Table IV and Figs. 1—3 present the facts obtained in this seriatim study concerning the manner and relative rapidity of growth of the body-characters with which we deal in this paper. The curves of the means are all of the same general type, with total body length showing far more rapid acceleration than the other body dimensions. Stem length, chest circumference and head circumference are similar in size and shape of curve throughout, though the latter shows smaller increments. Of the transverse measures, the shoulders are largest and grow the most. Next in size come hip width, chest width, and finally chest depth. Weight cannot be compared directly with these measures, but its curve is more nearly like that of height than any of the others.

Figs. 4—7 present a comparison of this sample with other samples previously reported upon (41). The data for the Growth Study, as already noted, are seriatim; all the other samples are cross-sectional. As was to be expected in the light of Woodbury's data, which showed Californian infants excelling those from other sections of the country in height and in weight, our sample, especially of the boys, is seen to average heavier and taller from the sixth month on than the samples from Iowa, New York City, and the United States at large. The Growth Study sample also averages somewhat taller and heavier than does the Californian sample above mentioned, except during the first six or eight months. During the first half-year the growth curves of all four groups are almost identical.

TABLE IV.

Table of Means, S.D.'s, P.E.'s of Means and Coefficients of Variation of nine measures made repeatedly on the same Infants from 1 *through* 36 *months.*

(Centimetres and kilograms.)

Month		Length					Stem Length					Head Circumference				
		No.	*M*	P.E.$_M$	σ	*V*	No.	*M*	P.E.$_M$	σ	*V*	No.	*M*	P.E.$_M$	σ	*V*
1	Boys	24	55·45	·317	2·31	4·16	24	36·00	·158	1·15	3·18	24	38·56	·164	1·19	3·09
	Girls	26	53·42	·319	2·41	4·52	26	34·90	·224	1·69	4·85	26	37·13	·161	1·22	3·28
	Both	50	54·39	·213	2·23	4·11	50	35·42	·157	1·65	4·65	50	37·82	·129	1·35	3·57
2	Boys	31	58·86	·222	1·84	3·12	31	38·64	·267	2·21	5·71	31	40·11	·137	1·13	2·82
	Girls	27	56·98	·280	2·16	3·79	27	37·07	·185	1·43	3·86	27	38·82	·154	1·19	3·06
	Both	58	57·98	·201	2·27	3·91	58	37·91	·180	2·03	5·36	58	39·51	·117	1·32	3·33
3	Boys	31	61·73	·204	1·68	2·72	31	40·32	·143	1·18	2·92	31	41·54	·145	1·20	2·89
	Girls	30	59·72	·276	2·24	3·75	29	38·98	·187	1·49	3·83	29	40·20	·133	1·06	2·63
	Both	61	60·74	·193	2·24	3·69	60	39·67	·128	1·47	3·70	60	40·89	·118	1·35	3·30
4	Boys	30	64·11	·227	1·85	2·88	30	41·82	·130	1·06	2·52	30	42·74	·126	1·03	2·40
	Girls	29	62·37	·224	1·79	2·88	29	40·85	·215	1·72	4·20	29	41·40	·142	1·13	2·73
	Both	59	63·25	·184	2·09	3·31	59	41·34	·135	1·54	3·73	59	42·08	·114	1·29	3·07
5	Boys	30	66·53	·253	2·06	3·09	30	43·36	·183	1·49	3·43	30	43·93	·145	1·18	2·69
	Girls	27	64·57	·307	2·37	3·67	27	41·81	·245	1·89	4·52	27	42·45	·157	1·21	2·85
	Both	57	65·60	·218	2·44	3·72	57	42·62	·172	1·92	4·52	57	43·23	·124	1·39	3·21
6	Boys	30	68·48	·266	2·16	3·16	30	44·59	·186	1·51	3·38	30	44·93	·144	1·17	2·61
	Girls	28	66·38	·272	2·13	3·21	28	43·18	·187	1·47	3·40	28	43·28	·148	1·16	2·67
	Both	58	67·46	·219	2·48	3·67	58	43·91	·145	1·64	3·73	58	44·14	·117	1·32	3·00
7	Boys	26	69·85	·309	2·34	3·35	26	45·36	·210	1·59	2·97	27	45·65	·172	1·33	2·91
	Girls	25	68·04	·357	2·65	3·89	24	44·19	·281	2·04	4·62	25	44·33	·178	1·32	2·98
	Both	51	68·96	·254	2·69	3·90	50	44·80	·181	1·90	4·23	52	45·02	·132	1·41	3·13
8	Boys	25	71·82	·333	2·47	3·43	25	46·52	·252	1·87	4·01	25	46·46	·175	1·29	2·78
	Girls	28	69·68	·355	2·78	3·99	28	45·11	·236	1·85	4·10	27	44·98	·157	1·21	2·69
	Both	53	70·69	·263	2·84	4·01	53	45·78	·179	1·93	4·21	52	45·69	·138	1·48	3·24
9	Boys	27	72·89	·297	2·28	3·13	27	47·36	·223	1·71	3·62	27	47·04	·150	1·15	2·45
	Girls	29	70·34	·314	2·51	3·56	29	45·43	·200	1·59	3·51	29	45·35	·162	1·29	2·85
	Both	56	71·57	·244	2·71	3·79	56	46·36	·173	1·92	4·14	56	46·16	·141	1·56	3·39
10	Boys	28	74·28	·287	2·25	3·03	28	47·90	·218	1·71	3·56	26	47·65	·168	1·27	2·67
	Girls	27	71·67	·327	2·52	3·58	27	46·18	·230	1·78	3·84	22	45·84	·190	1·32	2·88
	Both	55	73·00	·246	2·70	3·70	55	47·06	·172	1·89	4·01	48	46·82	·154	1·58	3·38
11	Boys	27	75·57	·333	2·57	3·40	27	48·84	·213	1·64	3·36	25	47·93	·153	1·14	2·37
	Girls	23	73·23	·388	2·76	3·77	23	46·94	·272	1·94	4·12	18	46·53	·242	1·52	3·27
	Both	50	74·49	·281	2·95	3·96	50	47·97	·188	1·98	4·11	43	47·34	·159	1·54	3·26
12	Boys	28	76·89	·323	2·53	3·30	28	49·16	·242	1·90	3·86	28	48·50	·165	1·30	2·68
	Girls	28	74·49	·335	2·63	3·53	28	47·66	·240	1·88	3·95	27	46·75	·176	1·36	2·90
	Both	56	75·69	·257	2·85	3·76	56	48·41	·184	2·04	4·21	55	47·64	·146	1·60	3·37
15	Boys	27	80·55	·370	2·85	3·53	27	51·23	·238	1·83	3·58	27	49·31	·179	1·38	2·80
	Girls	25	77·66	·372	2·76	3·55	25	49·00	·242	1·80	3·67	25	47·52	·148	1·10	2·32
	Both	52	79·16	·296	3·16	3·99	52	50·16	·197	2·11	4·20	52	48·45	·143	1·53	3·16
18	Boys	23	83·18	·470	3·34	4·02	22	52·09	·255	1·77	3·40	22	49·82	·187	1·30	2·62
	Girls	24	80·89	·393	2·85	3·53	24	49·90	·248	1·80	3·61	23	47·98	·181	1·28	2·68
	Both	47	82·01	·326	3·31	4·04	46	50·95	·205	2·06	4·05	45	48·88	·159	1·58	3·23
24	Boys	24	88·44	·452	3·28	3·71	24	53·50	·257	1·87	3·49	24	50·73	·197	1·43	2·83
	Girls	19	86·08	·313	2·02	2·35	19	51·18	·185	1·20	2·34	18	49·03	·184	1·15	2·36
	Both	43	87·40	·308	2·99	3·42	43	52·48	·196	1·91	3·63	42	50·00	·165	1·59	3·18
30	Boys	25	93·05	·559	4·14	4·45	25	54·19	·259	1·92	3·54	25	51·02	·179	1·33	2·60
	Girls	21	90·71	·468	3·18	3·51	21	52·66	·295	2·00	3·80	21	49·26	·180	1·22	2·48
	Both	46	91·98	·391	3·93	4·28	46	53·49	·211	2·12	3·96	46	50·22	·149	1·49	2·97
36	Boys	25	97·38	·588	4·36	4·48	25	55·68	·316	2·34	4·21	23	51·30	·187	1·33	2·59
	Girls	24	95·02	·457	3·32	3·49	24	53·46	·239	1·74	3·26	24	49·72	·143	1·04	2·08
	Both	49	96·22	·396	4·11	4·27	49	54·59	·229	2·38	4·36	47	50·49	·146	1·49	2·94

TABLE IV—(*continued*).

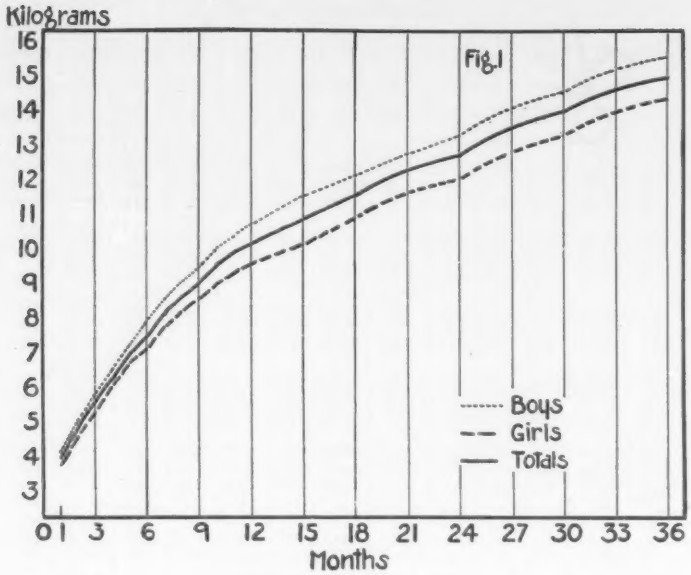
(Centimetres and kilograms.)

Month		Shoulder Width					Chest Width					Chest Depth				
		No.	*M*	P.E.$_M$	σ	*V*	No.	*M*	P.E.$_M$	σ	*V*	No.	*M*	P.E.$_M$	σ	*V*
1	Boys	23	14·86	·091	·65	4·38	22	10·46	·111	·77	7·38	24	9·89	·083	·60	6·07
	Girls	26	14·10	·082	·62	4·38	25	10·15	·101	·75	7·37	26	9·70	·081	·61	6·33
	Both	49	14·46	·068	·71	4·88	47	10·30	·073	·74	7·21	50	9·79	·059	·62	6·38
2	Boys	31	15·94	·092	·76	4·75	31	11·08	·127	1·05	9·47	31	10·35	·080	·66	6·4
	Girls	27	14·81	·143	1·10	7·45	25	10·88	·091	·67	6·16	27	10·21	·090	·69	6·77
	Both	58	15·41	·099	1·12	7·26	56	10·99	·082	·91	8·28	58	10·28	·063	·71	6·95
3	Boys	30	16·75	·072	·58	3·47	30	11·97	·139	1·13	9·44	30	10·57	·082	·67	6·34
	Girls	28	16·12	·082	·64	4·00	28	11·31	·091	·71	6·29	27	10·18	·088	·67	6·62
	Both	58	16·44	·067	·76	4·60	58	11·65	·090	1·02	8·72	57	10·38	·066	·74	7·10
4	Boys	30	17·46	·091	·74	4·25	30	11·89	·118	·96	8·08	30	10·68	·074	·60	5·59
	Girls	29	17·10	·111	·89	5·19	29	12·12	·108	·86	7·10	29	10·60	·101	·81	7·64
	Both	59	17·28	·076	·87	5·02	59	12·01	·077	·87	7·28	59	10·64	·063	·72	6·75
5	Boys	30	18·25	·134	1·09	5·98	30	12·76	·121	·98	7·68	30	11·01	·059	·48	4·32
	Girls	26	17·83	·151	1·15	6·42	26	12·36	·076	·58	4·66	26	10·72	·070	·53	4·93
	Both	56	18·05	·106	1·17	6·49	56	12·57	·078	·87	6·95	56	10·87	·052	·58	5·29
6	Boys	30	18·98	·117	·95	4·99	30	13·26	·133	1·08	8·15	29	11·11	·088	·70	6·29
	Girls	28	18·43	·148	1·16	6·32	28	12·88	·119	·93	7·26	28	10·87	·077	·61	5·59
	Both	58	18·72	·093	1·05	5·58	58	13·08	·084	·94	7·21	57	10·99	·061	·68	6·22
7	Boys	27	19·46	·128	·99	5·08	27	13·67	·102	·79	5·76	26	11·32	·085	·64	5·66
	Girls	24	19·01	·149	1·08	5·68	24	13·45	·118	·86	6·39	25	10·96	·095	·71	6·43
	Both	51	19·25	·098	1·04	5·41	51	13·57	·076	·80	5·90	51	11·15	·061	·64	5·77
8	Boys	25	19·91	·165	1·23	6·15	25	14·26	·138	1·02	7·15	25	11·35	·056	·41	3·64
	Girls	26	19·43	·129	·98	5·02	26	13·81	·115	·87	6·31	26	11·37	·113	·86	7·52
	Both	51	19·66	·111	1·18	5·98	51	14·03	·092	·98	6·97	51	11·36	·064	·68	5·97
9	Boys	27	20·33	·142	1·09	5·37	27	14·35	·125	·96	6·69	26	11·57	·082	·62	5·34
	Girls	29	19·64	·144	1·15	5·84	29	14·06	·115	·92	6·53	29	11·45	·092	·74	6·42
	Both	56	19·97	·108	1·19	5·98	56	14·20	·086	·95	6·68	55	11·51	·060	·66	5·71
10	Boys	25	20·74	·151	1·12	5·39	25	14·76	·126	·94	6·33	26	11·98	·086	·65	5·41
	Girls	22	19·93	·181	1·26	6·32	22	14·18	·149	1·04	7·33	22	11·67	·090	·62	5·34
	Both	47	20·36	·124	1·26	6·18	47	14·49	·100	1·02	7·01	48	11·84	·043	·44	3·75
11	Boys	24	21·00	·149	1·08	5·14	24	14·86	·121	·88	5·92	26	11·96	·082	·62	5·16
	Girls	19	20·25	·199	1·29	6·37	19	14·73	·127	·82	5·57	19	11·59	·100	·65	5·59
	Both	43	20·67	·126	1·22	5·91	43	14·80	·090	·88	5·93	45	11·81	·060	·60	5·06
12	Boys	25	21·09	·141	1·04	4·94	28	15·10	·120	·94	6·24	28	12·17	·088	·69	5·67
	Girls	26	20·47	·121	·91	4·45	27	14·90	·093	·72	4·83	28	11·88	·064	·50	4·20
	Both	51	20·77	·105	1·11	5·35	55	15·00	·079	·86	5·75	56	12·02	·060	·67	5·54
15	Boys	27	21·48	·172	1·33	6·18	27	15·98	·120	·93	5·79	26	12·29	·086	·65	5·28
	Girls	25	20·71	·124	·92	4·42	25	15·28	·089	·66	4·31	25	12·11	·083	·62	5·09
	Both	52	21·11	·113	1·21	5·73	52	15·64	·085	·91	5·82	51	12·20	·062	·66	5·38
18	Boys	22	22·18	·154	1·07	4·81	22	16·06	·123	·86	5·34	22	12·48	·124	·86	6·90
	Girls	22	21·40	·124	·86	4·03	22	15·53	·116	·81	5·19	23	12·35	·093	·66	5·32
	Both	44	21·79	·107	1·05	4·80	44	15·79	·093	·92	5·81	45	12·42	·072	·71	5·73
24	Boys	23	22·59	·163	1·15	5·11	22	16·45	·144	1·00	6·10	23	12·77	·115	·82	6·41
	Girls	18	21·99	·165	1·04	4·73	19	15·92	·105	·68	4·24	19	12·17	·114	·74	6·05
	Both	41	22·33	·117	1·11	4·97	41	16·20	·099	·95	5·84	42	12·50	·086	·83	6·62
30	Boys	25	22·51	·218	1·62	7·18	24	16·96	·115	·83	4·90	25	12·68	·148	1·10	8·66
	Girls	21	22·20	·130	·88	3·97	21	16·26	·088	·60	3·69	21	12·21	·128	·87	7·12
	Both	46	22·37	·132	1·33	5·94	45	16·63	·085	·85	5·08	46	12·46	·107	1·07	8·59
36	Boys	25	23·54	·132	·98	4·16	23	17·20	·122	·87	5·05	23	13·00	·127	·90	6·93
	Girls	24	23·08	·123	·89	3·87	24	16·55	·096	·69	4·19	24	12·42	·083	·61	4·87
	Both	49	23·32	·089	·92	3·94	47	16·87	·082	·83	4·93	47	12·70	·084	·85	6·68

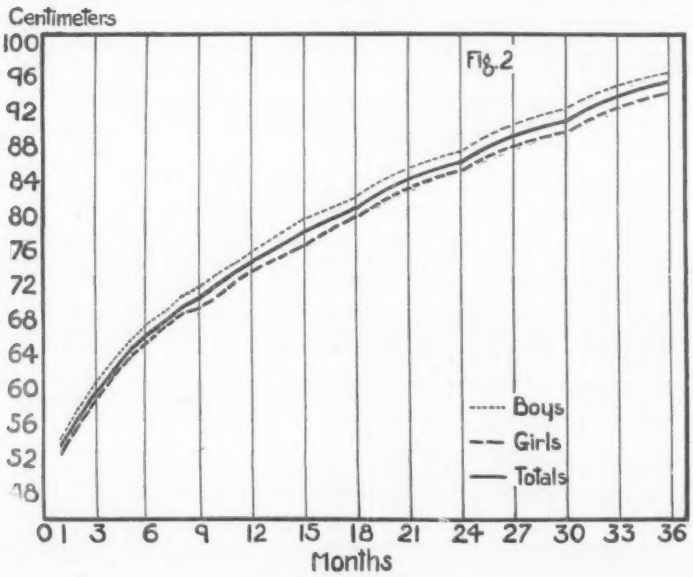
TABLE IV—(*continued*).

(Centimetres and kilograms.)

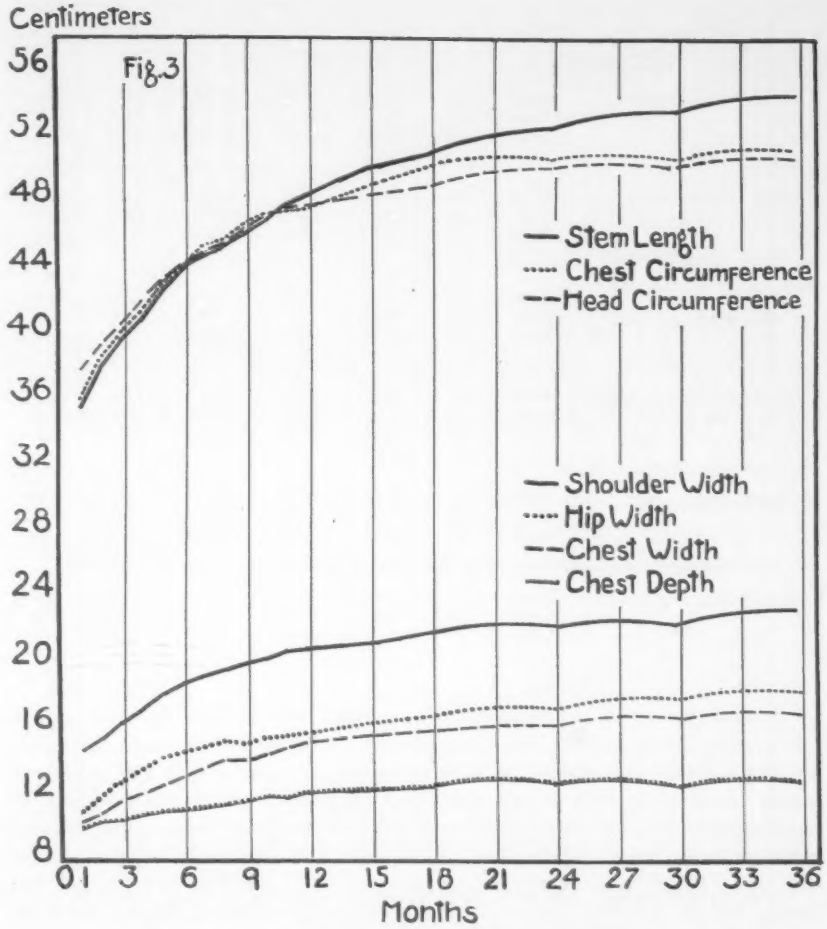
Month		Chest Circumference					Hip Width					Weight				
		No.	*M*	P.E.$_M$	σ	*V*	No.	*M*	P.E.$_M$	σ	*V*	No.	*M*	P.E.$_M$	σ	*V*
1	Boys	24	36·49	·231	1·68	4·61	23	11·03	·111	·79	7·19	25	4·37	·065	·44	4·61
	Girls	26	35·43	·210	1·59	4·48	26	10·62	·119	·90	8·52	26	3·98	·080	·53	6·13
	Both	50	35·94	·163	1·70	4·74	49	10·82	·080	·83	7·67	51	4·17	·055	·52	5·67
2	Boys	31	38·95	·231	1·91	4·90	31	12·06	·114	·94	7·80	31	5·28	·06	·50	4·29
	Girls	27	37·87	·198	1·53	4·04	27	11·91	·124	·96	8·05	27	4·83	·08	·63	5·96
	Both	58	38·45	·159	1·80	4·67	58	11·99	·084	·95	7·95	58	5·08	·05	·59	5·31
3	Boys	30	40·44	·227	1·85	4·56	30	12·94	·137	1·12	8·62	31	6·05	·08	·66	4·91
	Girls	29	39·68	·229	1·83	4·60	28	12·84	·128	1·01	7·85	30	5·51	·08	·68	5·58
	Both	59	40·07	·161	1·84	4·58	58	12·90	·090	1·02	7·87	61	5·78	·06	·71	5·62
4	Boys	30	41·89	·262	2·13	5·08	30	13·58	·116	·94	6·93	30	6·73	·08	·66	4·47
	Girls	29	41·10	·192	1·53	3·73	29	13·48	·144	1·15	8·54	29	6·24	·09	·72	5·33
	Both	59	41·50	·169	1·92	4·63	59	13·53	·093	1·06	7·81	59	6·49	·07	·74	5·24
5	Boys	29	43·44	·238	1·90	4·38	29	14·18	·125	1·00	7·02	31	7·49	·10	·77	4·71
	Girls	27	42·67	·261	2·01	4·71	26	14·07	·154	1·17	8·28	27	6·92	·11	·84	5·54
	Both	56	43·07	·179	1·98	4·60	55	14·13	·097	1·07	7·56	58	7·22	·08	·85	5·40
6	Boys	30	44·40	·205	1·67	3·76	29	14·81	·137	1·09	7·39	31	8·12	·09	·76	4·23
	Girls	28	43·84	·321	2·52	5·75	28	14·29	·131	1·03	7·18	28	7·38	·115	·88	5·48
	Both	58	44·13	·189	2·14	4·85	57	14·55	·100	1·12	7·71	59	7·76	·08	·91	5·31
7	Boys	27	46·08	·282	2·17	4·71	27	15·15	·127	·98	6·47	27	8·76	·09	·72	3·77
	Girls	25	44·69	·286	2·12	4·75	24	14·65	·168	1·22	8·32	26	7·99	·13	·95	5·42
	Both	52	45·41	·170	1·82	4·00	51	14·92	·103	1·09	7·32	53	8·38	·08	·93	4·97
8	Boys	25	46·74	·258	1·91	4·09	25	15·48	·124	·92	5·93	25	9·30	·115	·83	4·08
	Girls	26	45·64	·266	2·01	4·41	25	15·21	·159	1·18	7·76	28	8·43	·13	·99	5·36
	Both	51	46·18	·192	2·03	4·40	50	15·34	·105	1·10	7·18	53	8·85	·09	1·02	5·25
9	Boys	27	47·34	·226	1·74	3·67	26	15·48	·103	·78	5·04	27	9·75	·10	·76	3·58
	Girls	29	46·23	·274	2·19	4·73	29	14·89	·167	1·33	8·96	29	8·77	·08	·64	3·35
	Both	56	46·77	·181	2·00	4·28	55	15·17	·104	1·14	7·51	56	9·23	·10	1·05	5·18
10	Boys	25	47·86	·259	1·92	4·02	25	15·78	·120	·89	5·65	28	10·28	·115	·88	3·89
	Girls	23	46·83	·306	2·18	4·66	22	15·08	·192	1·33	8·83	28	9·17	·14	1·04	5·19
	Both	48	47·36	·213	2·19	4·61	47	15·45	·117	1·19	7·68	56	9·72	·10	1·12	5·25
11	Boys	24	48·31	·275	2·00	4·14	24	15·98	·120	·87	5·45	27	10·63	·13	·99	4·29
	Girls	19	46·71	·311	2·02	4·31	19	15·16	·180	1·16	7·66	24	9·53	·15	1·07	5·12
	Both	43	47·60	·225	2·19	4·60	43	15·62	·110	1·07	6·86	51	10·11	·11	1·17	5·29
12	Boys	27	48·66	·295	2·27	4·66	26	16·24	·128	·97	5·95	28	10·96	·14	1·04	4·34
	Girls	26	46·68	·221	1·67	3·58	26	15·44	·139	1·05	6·81	28	9·80	·15	1·13	5·25
	Both	53	47·69	·205	2·21	4·64	52	15·84	·102	1·09	6·86	56	10·38	·114	1·22	5·36
15	Boys	27	49·85	·308	2·37	4·76	27	16·79	·136	1·05	6·25	27	11·77	·15	1·13	4·35
	Girls	25	48·06	·280	2·08	4·33	25	15·92	·147	1·09	6·83	27	10·39	·16	1·19	5·25
	Both	52	48·99	·225	2·40	4·90	52	16·37	·169	1·16	7·11	54	11·08	·125	1·34	5·34
18	Boys	22	50·60	·358	2·49	4·92	22	17·37	·138	·96	5·52	26	12·38	·17	1·24	4·57
	Girls	22	48·80	·269	1·87	3·84	22	16·18	·143	·99	6·12	25	11·14	·17	1·24	5·25
	Both	44	49·70	·242	2·38	4·79	44	16·78	·112	1·11	6·59	51	11·77	·14	1·34	5·37
24	Boys	23	51·31	·294	2·09	4·06	22	17·45	·152	1·05	6·04	25	13·53	·22	1·91	5·44
	Girls	18	49·41	·333	2·10	4·24	18	16·85	·155	·97	5·77	22	12·22	·17	1·16	4·32
	Both	41	50·48	·237	2·25	4·45	40	17·19	·104	·98	5·68	47	12·92	·15	1·56	5·49
30	Boys	25	51·32	·188	1·40	2·72	25	17·87	·102	·76	4·25	25	14·75	·23	1·65	5·14
	Girls	21	49·53	·250	1·70	3·43	21	17·35	·146	·99	5·72	21	13·55	·19	1·35	4·54
	Both	46	50·50	·233	2·34	4·64	46	17·63	·093	·94	5·31	46	14·21	·16	1·63	5·27
36	Boys	25	52·09	·310	2·30	4·42	25	18·36	·119	·89	4·83	25	15·78	·24	1·80	5·22
	Girls	24	50·15	·288	2·09	4·17	24	17·86	·130	·94	5·28	24	14·52	·18	1·35	4·24
	Both	49	51·14	·232	2·40	4·70	49	18·11	·096	1·00	5·50	49	15·17	·16	1·72	5·17



Curves of Growth in Weight.



Curves of Growth in Length.



Curves of Growth in Seven Body Dimensions (Sexes Combined).

TABLE V.

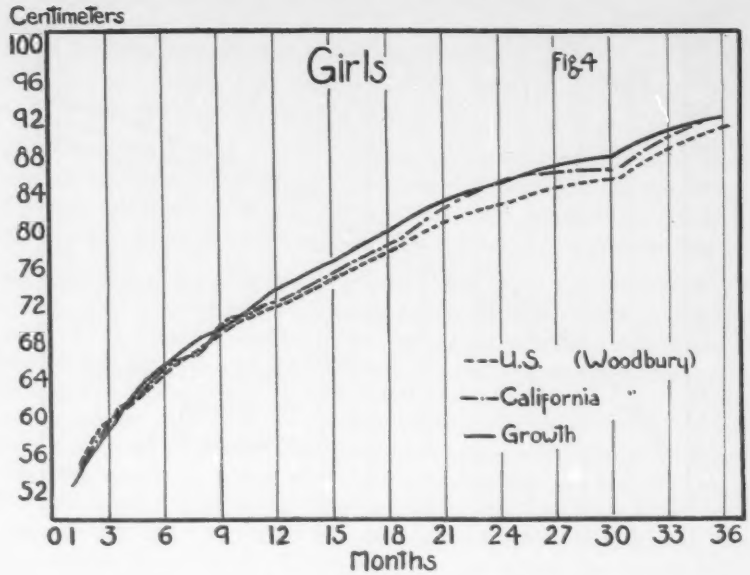
Birth Measurements.

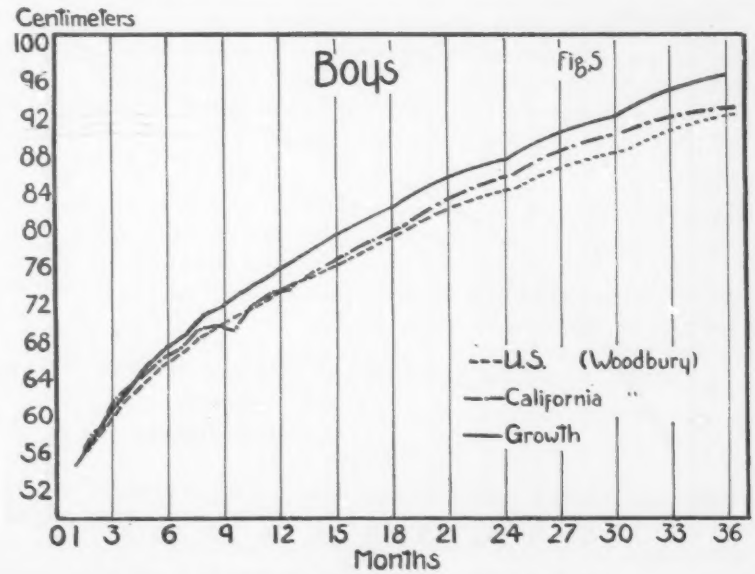
Measure	Sex	No.	Mean	Median	S.D.	Range
Head Circumference	Boys	28	35·52	35·40	1·1499	33·3 to 39·0
	Girls	25	34·52	34·50	1·3778	31·3 to 37·0
	Both	53	35·05	35·00	1·8801	31·3 to 39·0
Chest Circumference	Boys	28	33·50	33·45	1·3919	31·0 to 36·3
	Girls	24	33·51	33·75	2·1703	29·5 to 38·5
	Both	52	33·50	33·55	1·8365	29·5 to 38·5
Transverse Head Diameter	Boys	28	9·53	9·50	·6559	8·6 to 10·4
	Girls	25	9·23	9·20	·5709	8·2 to 10·4
	Both	53	9·39	9·40	·4884	8·2 to 10·4
Anterior-Posterior Head Diameter	Boys	28	12·28	12·30	·5283	11·3 to 13·4
	Girls	25	11·90	12·00	·6242	10·4 to 13·3
	Both	53	12·10	12·10	·6131	10·4 to 13·4
Weight (lb.)	Boys	31	8·17	8·19	·6971	6·75 to 9·63
	Girls	30	7·37	7·19	1·2828	5·38 to 10·56
	Both	61	7·78	7·88	1·1024	5·38 to 10·56
Weight (kg.)	Boys	31	3·70	3·71	·32	3·06 to 4·36
	Girls	30	3·35	3·27	·58	2·44 to 4·79
	Both	61	3·52	3·58	·50	2·44 to 4·79

A comparison with the measures of the Iowa Child Welfare Research Station (26) is especially interesting, since the same measurements were made, and at the same ages, in the two studies. Also, the number of cases for each age is approximately the same for both groups. The Iowa data contain many remeasures of the same children, but since the samples were not limited to children on whom seriatim measures were made, the samples for the different ages are essentially cross-sectional. The growth curves for all measures are closely similar for the two groups, except that for reasons already considered, the Growth Study curves are comparatively very smooth.

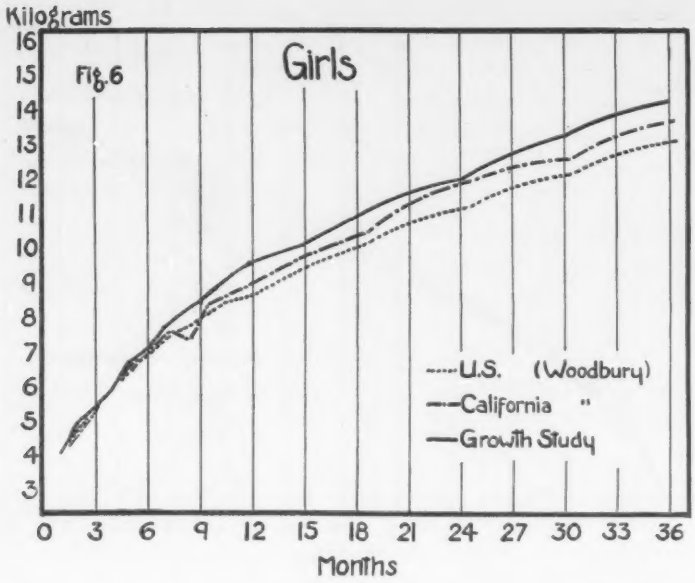
There is very little difference in the size of these two groups, except that the California babies usually measure slightly larger after six months. The greatest difference is in weight.

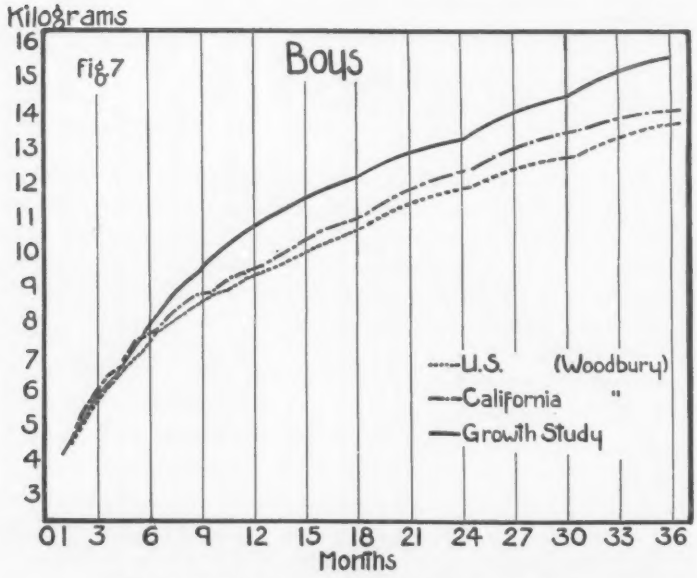
The mean weight and height of approximately 200 Berkeley infants (a representative sample) for months 12, 21 and 24 are almost identical with those of the smaller group of Berkeley infants we have been considering. The mean weights





Figs. 4 and 5. Comparative Curves of Growth in Length.





Figs. 6 and 7. Comparative Curves of Growth in Weight.

of the larger group of infants are 10·52, 12·30 and 13·11 kg. for 12, 21 and 24 months, respectively. For these same ages their mean lengths are 75·18, 84·84 and 87·88 cm. Whatever selective factors are operating to make the Growth Study infants taller and heavier than infants in other parts of the country, are also effective for the population of Berkeley as a whole.

When the growth trends of the various dimensions are made more directly comparable with each other by converting each measure into the percentage of itself at one month (Figs. 8—11), the measure which far outstrips all others is weight. This, of course, is to be expected, as it is a volumetric measure.

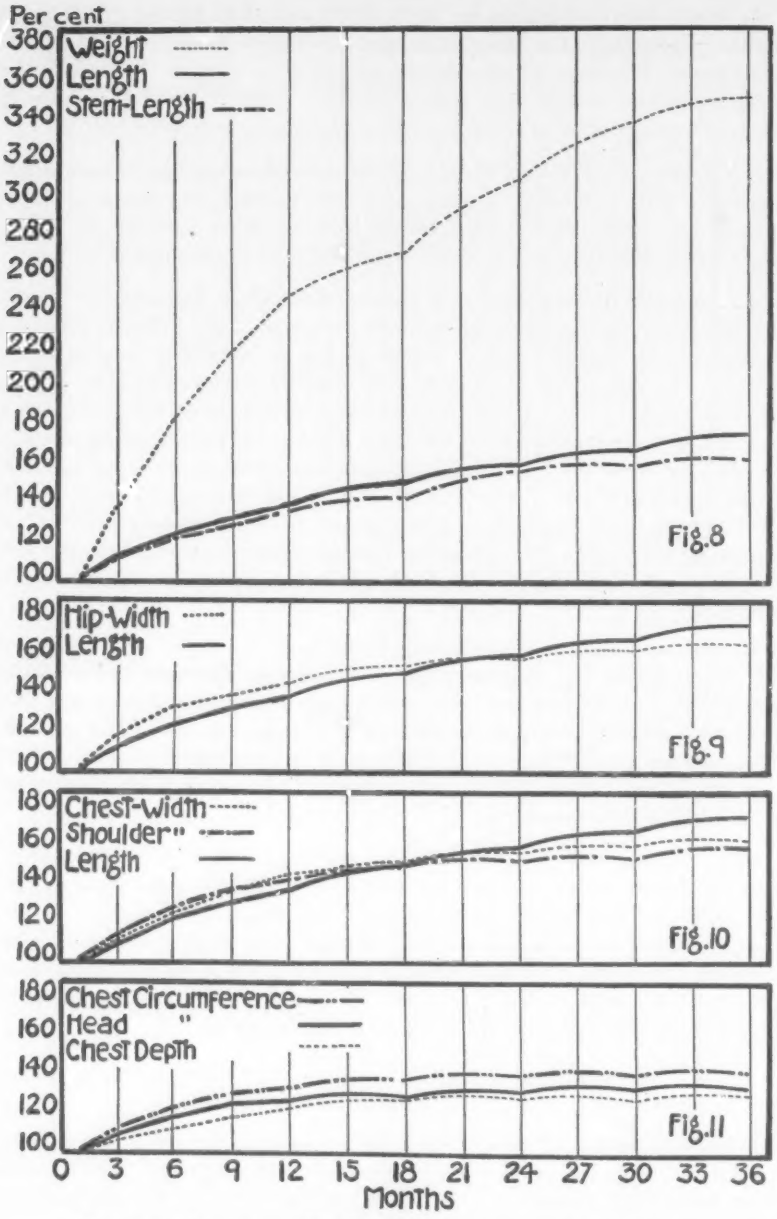
An inspection of the proportional growth curves of the linear measures reveals very interesting differences in the rates of increment of the different dimensions. Throughout the first 18 months (when growth in weight is most rapid) the measures which show the greatest increase are width measures—hips, shoulders, and chest—and it is not until the 24th-month records that we find the increments for height greater than those for the three widths. As for the height measures, the first year shows practically equal proportionate growth for the stem length and for the total length. It is only after the 12th month that the legs begin to grow rapidly and cause a relatively greater increase in the total height curve. The dimension which shows the smallest percentage increment throughout the entire 36 months, is chest depth. Head circumference shows the second smallest increment.

In general, though not consistently throughout, the cephalo-caudal direction of growth is followed in the percentage growth curves. The head size is relatively more mature at birth than are other measures and its postnatal growth is small; in the early months growth in shoulder width is relatively more rapid than chest width, and this position is reversed later; stem and leg length grow at about the same rate at first, but after the first 18 months, leg length is the most rapidly increasing dimension measured.

The changing body proportions, then, are for the first year due mainly to relatively greater growth in transverse widths, and relatively small growth in head size; while for the second and third years, the decrease in growth rate of all dimensions except total height results in changed body proportions due to rapidly increasing leg length. We find, especially during the first year, not only changes in the relative length of body parts, but also in the relative widths; and indices of body build should be formulated with this fact in mind.

C. VARIABILITY.

When we consider the variation from the mean curves of growth, the coefficients of variation (Table IV, pp. 37—39) show no age changes for any of the measures except two. Chest width and hip width decrease slightly but consistently in variation, during the three-year period.



Figs. 8—11. Percentages of First Month's Measures achieved at Later Ages.

There are no significant sex differences in variation. Although the coefficients of the girls are slightly larger in ninety-three comparisons, while the boys are larger in only fifty-seven comparisons, none of the differences is significant. The only measure in which the boys V's are greater for most ages is chest width, and since this one measure shows a contrary tendency to all the others in this respect, it may possibly be indicative of a real sex difference. There is also a slight suggestion that the girls are possibly becoming more homogeneous in length and in chest depth, toward the end of this three-year period.

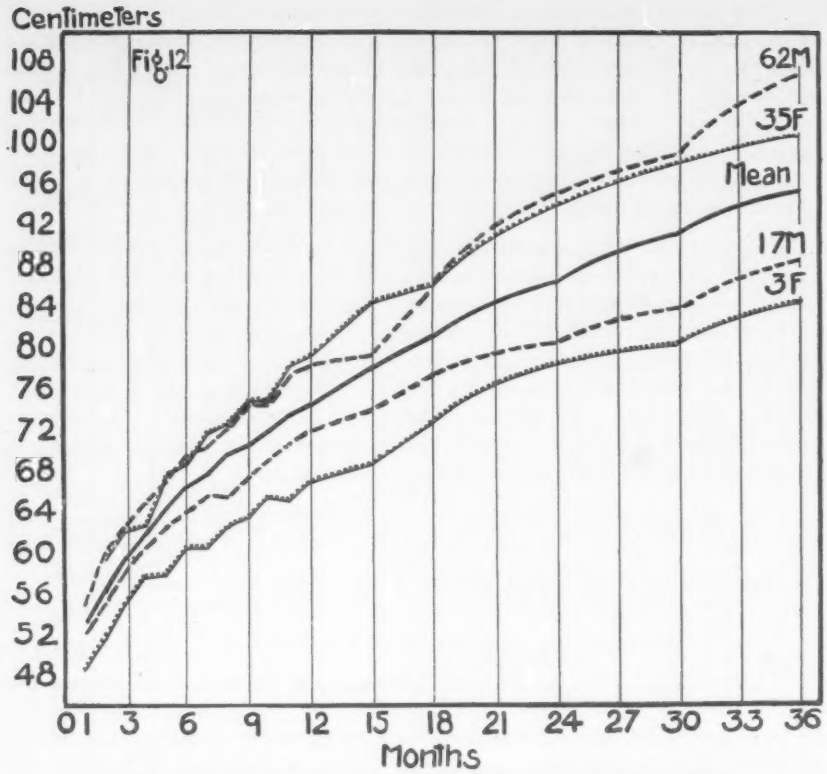
Montague and Hollingworth (32) found no tendency for sex differences in variability for seven measurements made on 2000 newborn infants. The measurements of the Iowa Child Welfare Research Station (26, 27) from one month through six years show no consistent sex differences in variation, with the possible exceptions of weight and hip width in the later ages. Both these studies do show, however, consistent tendencies for the boys to be larger. It seems unlikely, then, that any slight differences in the coefficient of variation found in the present group represent sex differences. The group is, however, typical of children, in general, so far as sex differences in size are concerned.

There are only slight tendencies for the standard deviations to increase with age, with the exception of weight, the standard deviation of which increases from ·54 kg. at month 1 to 1·72 kg. at month 36.

D. Consistency of Growth.

When we select, from the 36-month measures, the tallest boy and girl and the shortest boy and girl, and plot their individual curves of growth in length (Fig. 12), we find that the tall children have been consistently tall from the first month, and the short children consistently short. The extent to which this is true for the entire group in length and in other dimensions may be shown by correlation coefficients between different ages for the same dimension. Tables VI—XV give the correlations between the measures of these children at different ages. Head circumference, length and stem length are most consistent: the correlations between the 1-month and 36-month measures are, respectively, ·72, ·68 and ·59. Weight comes fourth, with correlations which are considerably lower, the r for the same interval being only ·40. There is, however, a marked increase in the consistency of this measure after the first year. The width measures of hip and shoulder diameters and chest circumference, equally low in consistency, all show the same phenomenon as does weight—a decidedly greater consistency after the first year. Chest width and depth, relatively unreliable measures, show low consistency and only small increase in consistency with growth.

The width measures become more consistent after their period of relatively rapid growth has stopped, and weight, which is largely influenced by the width measures, exhibits the same trend.



Individual Growth Curves of Tall and Short Children compared to the Mean.

Self-Correlations showing Consistency of Growth.

TABLE VI.

In Length.

Month	3	6	12	18	24	30	36
1	·87	·77	—	—	·68	—	·68
3	—	·84	—	—	·71	—	·67
6	—	—	·91	—	·83	—	·79
12	—	—	—	·91	·93	—	·88
18	—	—	—	—	·92	—	·89
24	—	—	—	—	—	·95	·92
30	—	—	—	—	—	—	·97

TABLE VII.

In Weight.

Month	3	6	12	18	24	30	36
1	·82	·54	—	—	·48	—	·40
3	—	·80	—	—	·52	—	·48
6	—	—	·72	—	·66	—	·61
12	—	—	—	·92	·91	—	·80
18	—	—	—	—	·93	—	·88
24	—	—	—	—	—	·94	·95
30	—	—	—	—	—	—	·96

TABLE VIII.

In Head Circumference.

Month	6	12	24	36
1	·79	·71	·74	·72
6	—	·86	·80	·81
12	—	—	·91	·93
24	—	—	—	·92

TABLE IX.

In Stem Length.

Month	6	12	24	36
1	·55	·53	·45	·59
6	—	·79	·71	·68
12	—	—	·86	·74
24	—	—	—	·75

TABLE X.

In Chest Circumference.

Month	6	12	24	36
1	·25	·29	·26	·29
6	—	·44	·41	·43
12	—	—	·73	·69
24	—	—	—	·82

TABLE XI.

In Chest Width.

Month	6	12	24	36
1	·14	·19	−·01	·23
6	—	·53	·37	·20
12	—	—	·54	·51
24	—	—	—	·71

TABLE XII.

In Chest Depth.

Month	6	12	24	36
1	·57	·40	·25	·12
6	—	·53	·51	·42
12	—	—	·45	·50
24	—	—	—	·63

TABLE XIII.

In Shoulder Width.

Month	6	12	24	36
1	·10	·33	·13	·21
6	—	·46	·33	·35
12	—	—	·68	·65
24	—	—	—	·71

TABLE XIV.

In Hip Width.

Month	6	12	24	36
1	·45	·37	·32	·23
6	—	·48	·60	·45
12	—	—	·80	·83
24	—	—	—	·83

TABLE XV.

Self-Correlations showing Consistency according to Sex.

Measure	Month								
	6 & 12			12 & 24			24 & 36		
	Boys	Girls	Total	Boys	Girls	Total	Boys	Girls	Total
Length	·89	·86	·91	·94	·88	·93	·93	·87	·92
Weight	·56	·78	·72	·89	·97	·91	·95	·92	·95
Hip Width	·33	·57	·48	·86	·70	·80	·84	·84	·83
W./H.^{2}	·53	·51	·44	·46	·68	·74	·76	·77	·73

The above correlations have all been computed for the total group, regardless of sex. The division into sex groups seemed unnecessary, since there were no significant differences in the growth tendencies of the two sexes. As a further check, however, consistency correlations were computed on the sexes separately for three measures, weight, length, and hip width, and the W./L.^{2} index, for three age intervals (Table XV). These correlations give no evidence of sex differences in consistency.

E. Interrelationships among the Measures.

Inter-Correlations between Individual Dimensions.

The correlations between the measures have been computed for representative ages. Table XVI gives these correlations, ranked from highest to lowest, in the order of the mean of the *r*'s between each pair. There is a tendency for the higher inter-correlations to be between weight and one of the more reliable width measures; and between the two length measures (which rank second and fourth in reliability). The three combinations which correlate most highly are: hip width and weight, with a mean *r* of ·84; stem length and total length, with a mean *r* of ·82; and chest circumference and weight, with a mean *r* of ·80. We find, also, fairly high relationships between stem length and weight, and total length and weight (·74). It is to be expected that weight should correlate highly with the various diameters and circumferences, since weight is a composite which includes them all, and since they are all positively correlated with each other.

TABLE XVI.

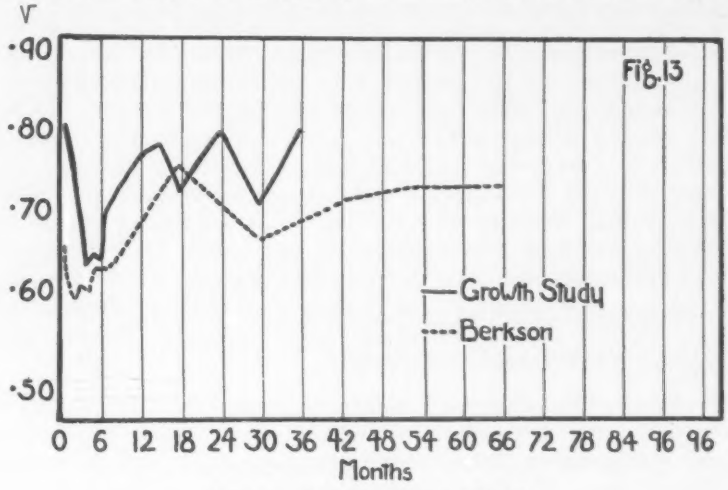
Inter-Correlations of Measures.

	Month									Mean
	1	3	6	9	12	15	18	24	36	
H.W. × W.	·896	·807	·816	·791	·821	·852	·875	·812	·854	·836
L. × S.L.	·742	·784	·810	·844	·879	·882	·797	·855	·791	·820
C.C. × W.	·746	·849	·791	·844	·792	·851	·857	·698	·770	·795
S.L. × W.	·729	·786	·685	·736	·782	·804	·707	·730	·732	·743
L. × W.*	·818	·708	·651	·734	·777	·795	·734	·808	·813	·739
S.W. × W.	·671	·748	·718	·718	·670	·696	·653	·707	·667	·694
C.C. × H.W.	·704	·711	·625	·706	·641	·732	·817	·652	·634	·691
S.W. × C.C.	·479	·697	·712	·691	·628	·700	·613	·604	·647	·641
S.W. × H.W.	·624	·586	·467	·613	·469	·635	·697	·611	·647	·594
L. × S.W.	·587	·670	·642	·562	·536	·570	·571	·658	·496	·588
L. × H.W.	·598	·489	·336	·491	·649	·655	·636	·666	·732	·584
H.W. × S.L.	·644	·579	·393	·402	·635	·716	·668	·584	·601	·580
C.C. × S.L.	·557	·627	·493	·607	·582	·678	·632	·502	·537	·579
L. × C.C.	·498	·546	·538	·648	·595	·634	·529	·669	·550	·572
S.W. × S.L.	·453	·587	·496	·496	·544	·628	·491	·513	·533	·527

* Additional correlations between length and weight have been computed. They are for months 2, 4, 5, 7, 8, 11 and 30 and are, in order, ·762, ·644, ·658, ·707, ·727, ·768, and ·719. These have been included in computing the mean of the length-weight correlations.

The r's of ·69 between chest circumference and hip width, and of ·64 between chest circumference and shoulder width, indicate a fair amount of symmetry in the transverse trunk measures, but by no means perfect symmetry. The length-width correlations are the lowest ones and fall in the last six places in the rank order table. These r's range from ·53 to ·59. One should expect, then, to find in this group tall slender infants and short chubby ones, as well as the more frequently occurring large, medium and small sizes of symmetrical proportions. In general, there is a clear tendency for all measures to be positively related. Large children are for the most part large in all their measures, and small children small in all their measures.

As for age trends in these inter-correlations, there is a tendency for some of the coefficients to decrease at and near six months. This is true of the correlations between weight and length, hip width and length, and between hip width



Correlations between Weight and Length.

and stem length. The same tendency is present—but less clear-cut—in the correlations between shoulder width and hip width. Some other coefficients tend to decrease slightly after month 18. Among these are the correlations between weight and chest circumference, length and chest circumference, hip width and chest circumference, and between length and stem length.

Boas and Wissler (12) contend that correlations between measures are high during periods of rapid growth, and conclude from this that when rapid growth occurs in one part of the body, it occurs also in other parts. Berkson (9) has computed the inter-correlations between height, weight and chest circumference for a large number of cases, and concludes that the alternating points of high and low correlations which he finds correspond with critical periods of growth. His data for the first six years are taken from Woodbury (41), and this part of his curve of length-weight correlations is given in Fig. 13 with the correlations obtained

for the present group. The Growth Study curve for length-weight correlations shows high r's at months 1 and 15, with a consistent drop from 1 to 4, and a rise from 7 to 15, followed by variations showing no directional trend between 15 and 36 months. These r's differ from the Woodbury data in being high at month 1, with a directional trend of decreasing r's from months 1 to 4*, and no significant trend toward decrement after month 15.

A number of other investigators have computed r's between body measures, and their findings may be compared with ours. Taylor's (36) correlations between weight and length of 240 newborn infants are ·79 for boys and ·81 for girls—very close to our one-month correlation of ·82.

The correlations between weight and length given by Gray and Ayres for year groups, starting at one year, fail, as ours do, to show a decreased r at three years. Instead there is a rise from one to three followed by a drop at five years.

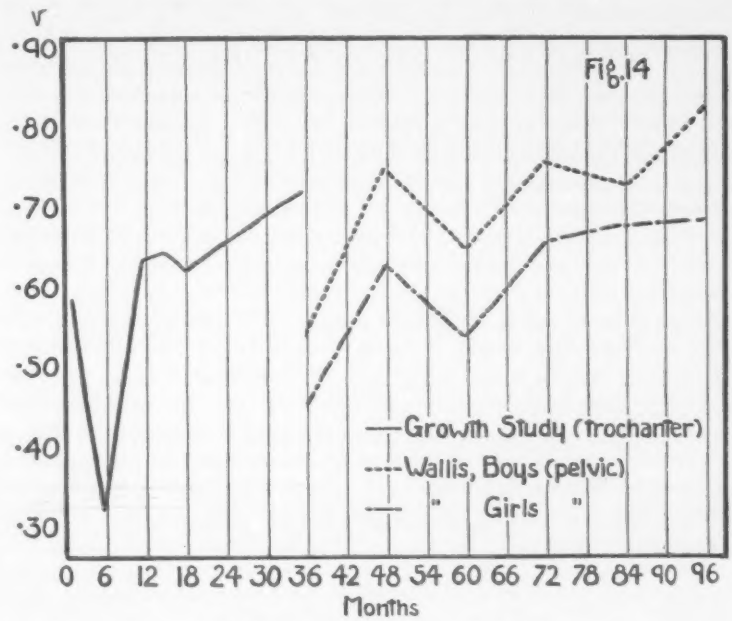
Baldwin (5) computed inter-correlations for about 120 children at each year from seven to seventeen years. He found that the correlations between different body measures were higher for the boys than for the girls, and that they were higher for both sexes in early adolescence, and lowest at 17 years. His length-weight correlations show trends very similar to those of Berkson. As has been already mentioned, Boas and Wissler (12) also find increased correlations during rapid pre-adolescent growth. Wallis (39) gives correlations between pelvic width and length of a group of New York boys from three to eight years. These are plotted in Fig. 14 with the length-hip width r's for the present group. Although the length-pelvic width r's of Wallis are smaller at three years than our length-trochanter width r's, they show the same directional trend of increasing correlation with age that is indicated in our sample after month 6.

The Woodbury measures, used by Berkson, though for large samples, were collected from many sources with the possibility of great differences in technique and reliability of the measures. In addition, it was necessary to compute the correlations from class groups in Woodbury's tables instead of from the original data. Since the measures of the Growth Study have been made by a single observer with a consistent method, and since the same children have been measured at the successive age levels, and since none of the other correlations in the studies cited show decreased r's at three years, this curve may well give the truer approximation of the changes in the body relations which occur during growth in the individual through the first three years.

So far, then, as the evidence for this short age period is concerned, any age differences in the inter-correlations of the body measures seem to be related to differences in the relative growth rates of the parts measured, and perhaps to individual differences in tendency to increased chubbiness during the first nine or

* There is some evidence for a similar trend in the Woodbury data, but in smoothing his curve Berkson does not accept this.

eleven months. Length-width correlations tend to drop from months 1 to 6 when growth in widths is more rapid and more variable among individuals, and then to rise again as the disparity in growth rates of the body parts involved diminishes. After eighteen months, the growth in chest circumference is very slow, and the correlations between it and the still rapidly growing measures—weight, length, hip width and stem length—tend to drop. For this sample, during the first three years, growth changes in correlation between the various body parts show more relation to the amount of disparity between rates of growth in these parts, than to successive periods of rapid and slow growth of the body as a whole. The situation may be



Correlations between Length and Hip Width.

very different at later ages when body proportions are changing less radically. But it is also possible that the high adolescent correlations between body measures are due to the *wider variation* of size at this time when some children start rapid growth at a younger age than do others. Boas (10) has found that, although there is greater variability in the size of adolescents than in older or younger children, "the maximum of variability of stature during adolescence disappears in groups having the same moments of maximum rate of growth." If the correlations were computed between measures on children of the same age of maximum rate of growth, rather than grouping together all children of the same age, the high adolescent r's would probably also disappear.

F. Body-build.

The recognised desirability of a quantitative measure of body-build which should reliably and adequately represent the empirically observed variations in human structure has been productive of much debate, and of a considerable amount of research as well, from the time of Quetelet up to the present. The evidence recently brought forward by Kretschmer, pointing to a close relationship between bodily proportions and temperamental predispositions, lends added significance to the the search for an objective index of build. Among the indices which have been proposed are the ratios of weight to height, weight to the square of height (Quetelet, Davenport), weight to the two and one-half power of height (Quetelet), weight to the cube of height (Rohrer and Bardeen), chest circumference to height (Davenport), hip width to height (Pryor and Lucas, Gray), stem length to height (Gray, Hejinian and Hatt), and many others. Objections have been raised to the use of an index which includes weight as one of the factors (Wertheimer and Hesketh (40)) on the ground that weight fluctuations may be of serious extent, even over a relatively small span of time, as a result of a variety of exogenous influences. This objection, however, while it may be of moment with subjects such as those used by Wertheimer and Hesketh (adult psychotics), may nevertheless be relatively inconsequential with subjects who are free from any serious ailment, either physical or behaviouristic. The self-correlations in Table I (p. 34 above) indicate that this objection is not important in the present sample. With infants and with children, at least, indices involving weight as one factor should not be dismissed on *a priori* grounds, because weight is one measure in the recording of which the personal equation plays a relatively minor rôle. We shall see that the superior reliability of measurements of body weight, with infant and child subjects such as those used in the present investigation, gives indices which are based in part upon weight measurements a definite advantage over various other proposed indices which incorporate measurements difficult to obtain, at these early ages, with a sufficiently high degree of accuracy.

Several different indices have been computed from the measures taken on the Growth Study children, and their means are given in Table XVII (p. 56) and Figs. 15—22. These curves exhibit interesting age trends in body proportions. As would be expected from the proportionate growth curves in Fig. 9, the relative hip width (Fig. 15) increases through the sixth month, but declines again, rapidly, after the seventh month. If we eliminate the leg measures by comparing hip width to stem length (Fig. 16) the initial rapid increase of hip width is still present, but after a short decrease between the seventh and ninth months, the relationship between the two measures remains constant. The ratio of stem length to total length, after a slow decrease through the ninth month, falls off very rapidly after this age. The same may be said of the ratios of chest circumference to length, and of chest circumference to stem length.

The length-weight indices vary greatly, according to the power of length used in computing the index. The simple W./L. ratio shows a growth curve of increasing

weight in relation to length, rapid at first, but slowing up in the second and third year. $W./L.^3$—a ratio which has been recommended on the ground that weight is a volumetric measure—shows a decreasing index which is rapid after the ninth month. $W./L.^2$—an index which has often been used—shows relatively increasing weight through the eleventh month, with a more gradual decrease after this age*. All three of these indices yield results on the present sample which are closely similar to age curves of the mean indices of other groups presented elsewhere. Davenport (14) gives, in tabular form, age changes in mean indices for these three† which are very similar to ours, and our $W./L.^2$ means are almost identical with those for the first three years derived from Quetelet's data (34).

[* At my suggestion the authors worked out the value of $I_d = W./(C.C. \times H.W. \times L.)$ as an index. It will be seen that this is an attempt to obtain weight divided by a product of three linear quantities, i.e. chest circumference, hip width and length. I did not suggest chest depth, as the authors considered it a not very reliable measure. Thus roughly we have an index of weight per volume of the body, or we have an index which seemed to me the nearest approach to what I should call "chubbiness"—a term not defined by the authors. Dr Bayley writes to me: "Chubbiness" as we used it in our ratings of photographs meant the tendency to be wide, relative to length, rather than to have excess fatty tissue, though this latter factor had a strong influence on the rating." (See below, p. 71.) The authors obtained the following results:

Index $W./(C.C. \times H.W. \times L.)$

	Age in months				
	1	5	12	18	36
Index	197	181	181	172	170

When these values are plotted it will be seen that the index decreased slowly with age, but in an almost *linear* manner. $I_d + \cdot 6431 \times$ (age in months) is a function which on the average is nearly constant, $= 189 \cdot 46$, for the first thirty-six months of life. It thus appears to be more satisfactory than several of those graphed in Figs. 15—22, the most nearly linear of which is probably $W./l^3$, an index which we might anticipate would have a fairly high correlation with the present index. I_d was found, however, to have *no* correlation with the judges' ratings of "chubbiness," nor with $W./l^3$, the index on which the writers lay stress. Now this latter index may be read as $W./l^3 \times l$, and this seems to indicate that a volumetric index of chubbiness should have no relation to length or absolute size of the infant, which I take it should be true for a real definition of "chubbiness." That this "index of chubbiness" has no relation to the judges' ratings of "chubbiness," I should, I am afraid, be inclined to account for by the fact that the judges have unconsciously paid considerable attention to size (l) in their estimate of "chubbiness," which accounts for the correlation between their ratings and the index $W./l^3 \times l = W./l^2$. Anyhow, if chubbiness equals plumpness, I think it should be practically independent of size, and that a reasonable scientific definition of it would be weight per unit of volume. Short of actually measuring the volume of the body by immersion—practically impossible with infants—we have to find rough measures of the volume of the body, or take a product of three linear dimensions which may reasonably be supposed proportional to it. I may have a different opinion from the authors of what is meant by "chubbiness," but I think at any rate we need a scientific definition of the term. I personally am inclined to identify it with average bodily density, and thus to free it from any *à priori* relation to size; we may afterwards show that, as is the case with the present index, density appears to decrease physiologically with age. For children of the same age, I should anticipate little relation of density to size. If there be, we may be running up against some other very important physiological fact. ED.]

† These tables are based on tables given by Bardeen (7), using the data of Schmid-Mounard and of Quetelet.

TABLE XVII.

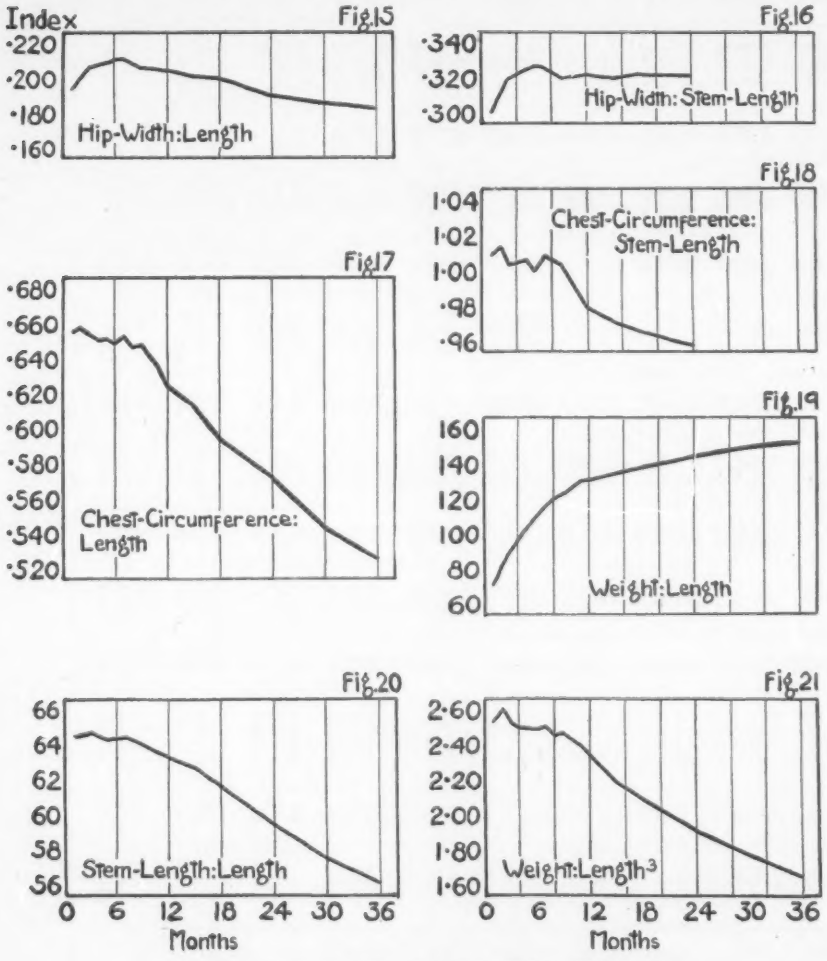
Table of Means for Boys, Girls, and Total Group of Eight Body-build Indices from 1 through 36 months.

Month		100 Weight / Length		Weight / Length²		Weight / Length³		100 Stem Length / Length		Chest Circumference / Length		Chest Circumference / Stem Length		Hip Width / Length		Hip Width / Stem Length	
		No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean
1	Boys	24	79·06	24	1·42	24	2·57	24	64·94	24	·658	24	1·014	23	·199	23	·307
	Girls	26	74·26	26	1·39	26	2·60	26	65·32	26	·664	26	1·017	26	·199	26	·304
	Both	50	76·56	50	1·41	50	2·59	50	65·14	50	·661	50	1·015	49	·199	49	·306
2	Boys	29	89·14	29	1·51	29	2·55	—	—	28	·660	31	1·016	—	—	—	—
	Girls	25	86·57	24	1·56	25	2·80	—	—	23	·668	27	1·022	—	—	—	—
	Both	54	87·95	53	1·53	54	2·66	—	—	51	·663	58	1·019	—	—	—	—
3	Boys	31	97·99	31	1·59	31	2·57	31	65·32	30	·655	30	1·004	30	·210	30	·321
	Girls	30	92·01	30	1·54	30	2·60	29	65·36	28	·665	29	1·015	28	·215	27	·329
	Both	61	95·05	61	1·56	61	2·59	60	65·34	58	·660	59	1·009	58	·212	57	·325
4	Boys	28	105·06	28	1·64	28	2·56	—	—	25	·653	—	—	—	—	—	—
	Girls	25	99·75	25	1·59	25	2·55	—	—	23	·661	—	—	—	—	—	—
	Both	53	102·56	53	1·62	53	2·56	—	—	48	·657	—	—	—	—	—	—
5	Boys	30	112·60	30	1·69	30	2·55	30	65·18	28	·653	28	1·001	28	·213	28	·326
	Girls	27	106·92	27	1·65	27	2·56	27	64·74	27	·661	27	1·021	26	·217	26	·336
	Both	57	109·91	57	1·67	57	2·55	57	64·97	55	·657	55	1·011	54	·215	54	·330
6	Boys	28	118·52	28	1·75	28	2·55	—	—	27	·647	30	·996	29	·217	29	·333
	Girls	24	111·53	24	1·68	24	2·54	—	—	22	·665	28	1·015	28	·215	28	·331
	Both	52	115·31	52	1·72	52	2·54	—	—	49	·655	58	1·005	57	·216	57	·332
7	Boys	26	124·45	26	1·78	26	2·56	26	64·95	26	·659	26	1·014	26	·217	26	·334
	Girls	25	117·58	25	1·73	25	2·54	24	65·06	24	·659	24	1·013	23	·215	23	·330
	Both	51	121·08	51	1·76	51	2·55	50	65·01	50	·659	50	1·014	49	·216	49	·332

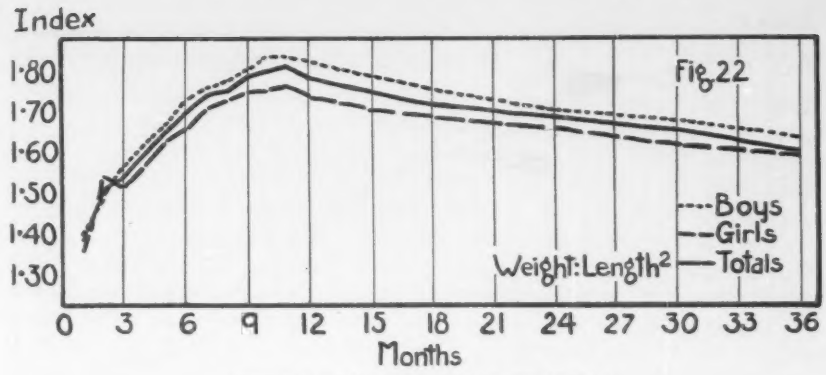
8	Boys	24	129·16	24	1·80	24	2·50	—	—	23	·649	—	—	—	—	—	—
	Girls	24	122·38	24	1·75	24	2·56	—	—	22	·657	—	—	—	—	—	—
	Both	48	125·77	48	1·77	48	2·51	—	—	45	·653	—	—	—	—	—	—
9	Boys	27	133·57	27	1·83	27	2·52	27	64·98	27	·650	27	1·000	26	·212	26	·327
	Girls	29	124·41	29	1·77	29	2·51	29	64·59	29	·657	29	1·018	29	·212	29	·328
	Both	56	128·83	56	1·80	56	2·52	56	64·78	56	·654	56	1·009	55	·212	55	·326
10	Boys	28	138·18	28	1·86	28	2·51	—	—	24	·644	—	—	—	—	—	—
	Girls	25	126·97	25	1·77	25	2·46	—	—	22	·653	—	—	—	—	—	—
	Both	53	132·89	53	1·82	53	2·49	—	—	46	·648	—	—	—	—	—	—
11	Boys	27	140·58	27	1·86	27	2·47	—	—	22	·642	—	—	—	—	—	—
	Girls	21	130·79	21	1·78	21	2·44	—	—	18	·639	—	—	—	—	—	—
	Both	48	136·30	48	1·83	48	2·45	—	—	40	·641	—	—	—	—	—	—
12	Boys	28	142·78	28	1·85	28	2·41	28	63·93	27	·633	27	·988	26	·212	26	·326
	Girls	28	131·18	28	1·76	28	2·37	28	64·00	26	·628	26	·981	26	·208	26	·328
	Both	56	136·98	56	1·81	56	2·39	56	63·97	53	·631	53	·985	52	·210	52	·327
15	Boys	27	146·12	27	1·81	27	2·25	27	63·62	27	·619	27	·973	27	·208	27	·328
	Girls	25	134·96	25	1·73	25	2·23	25	63·11	25	·619	25	·982	25	·205	25	·325
	Both	52	140·75	52	1·77	52	2·24	52	63·37	52	·619	52	·977	52	·207	52	·326
18	Boys	23	148·51	23	1·78	23	2·15	22	62·79	22	·608	21	·967	22	·209	21	·332
	Girls	24	138·11	24	1·71	24	2·13	24	61·70	23	·602	22	·975	22	·201	22	·324
	Both	47	143·20	47	1·74	47	2·14	46	62·22	45	·605	43	·971	44	·205	43	·328
24	Boys	23	153·80	23	1·73	23	1·96	24	60·51	23	·581	23	·962	22	·197	22	·325
	Girls	19	145·05	19	1·68	19	1·95	19	59·47	18	·574	18	·965	18	·196	18	·329
	Both	42	149·84	42	1·71	42	1·96	43	60·05	41	·578	41	·963	40	·196	40	·327
30	Boys	25	158·27	25	1·70	25	1·83	25	58·27	25	·552	—	—	25	·192	—	—
	Girls	21	149·13	21	1·64	21	1·81	21	58·06	21	·546	—	—	21	·191	—	—
	Both	46	154·10	46	1·67	46	1·82	46	58·18	46	·550	—	—	46	·192	—	—
36	Boys	25	161·75	25	1·66	25	1·71	25	57·21	25	·536	—	—	25	·189	—	—
	Girls	24	152·68	24	1·61	24	1·69	24	56·29	24	·528	—	—	24	·188	—	—
	Both	49	157·31	49	1·63	49	1·70	49	56·76	49	·532	—	—	49	·188	—	—

Most of these indices show rapid growth changes in body proportions, varying with the rates of growth of the parts compared. They are in accordance with our other findings, that early postnatal growth is most rapid in the transverse body dimensions, while after the first year leg length (and presumably arm length, which was not measured in this study) increases far more rapidly than any other body segment.

Any sex differences to be found in these body proportions are, again, very small, and with a sample of the size of the present study, certainly not significant. How-



Figs. 15—21. Curves of Growth in Seven Body-build Indices.



Curves of Growth in the Weight : Length² Index.

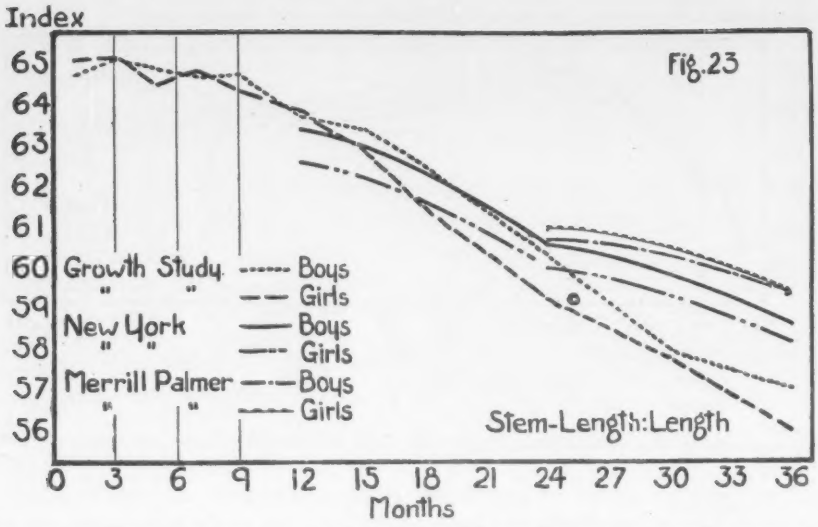
ever, so far as this group goes, during the first six months there is a tendency for the girls to have wider hips in proportion to total length and stem length (the latter is less clear); while after six months the situation is reversed, and the boys have relatively larger hips. The boys usually have a relatively longer stem length, when compared to stature, and after month 15 they have a greater chest circumference. When chest circumference is compared to stem length, however, the circumferences of the girls are larger. All of the weight-length indices show that, for the most part, the boys are heavier relative to their length.

Wallis (39) has computed recumbent stem length/length indices for New York private school children aged 1 to 8 years, and Merrill-Palmer school children, aged 2 to 5 years. Her indices, though in general their trends are similar to ours, decrease with age much less abruptly at the ages in which the Growth Study group was measured in the upright position (see Fig. 23, p. 60). The New York group shows the same sex difference as ours (higher indices for the boys) with the exception of years 7 and 8.

Davenport (15), following an extensive investigation of the statistical behaviour of the several indices cited above which are based upon weight and some power of height, concluded that either $W./H.^2$ or $W./H.^{2\frac{1}{2}}$ was to be preferred as an index of body build to either W./H. or the $W./H.^3$ index which Bardeen (7) had argued for on logical rather than empirical grounds. The two former indices yield smaller measures of dispersion than do the two latter, with a given group of subjects at a particular age, and they also fit better the empirical picture of developing body build.

Table XVIII (p. 60) adduces new evidence in support of the $W./H.^2$ index. (Since Quetelet's data showed both $W./H.^2$ and $W./H.^{2\frac{1}{2}}$ to be good* indices, with little to choose between them, the latter index was not computed for the data from the present

[* The authors assume here that a "good" index is one that correlates highly with the judges' ratings of "chubbiness." I should anticipate the reverse of this: see my footnote, p. 55. ED.]



Comparative Curves of the Stem Length : Length Index. (The New York and Merrill-Palmer Curves are based on Recumbent Measures (39). Growth Study Measures are recumbent through month 18.)

TABLE XVIII.

Correlations of Indices of Body-build with Judges' Ratings.

Index	Months											Mean r
	1	3	5	7	9 & 10	12	15	18	24	30	36	
$W./L.^2$	·75	·71	·83	·75	·84	·79	·77	·72	·55	·84	·79	·76
$W./L.^3$	·69	·68	·79	·79	·84	·74	·70	·53	·59	·71	·68	·71
W./L.	·66	·66	·72	·59	·66	·73	·62	·74	·59	·71	·66	·67
H.W./L.	·65	·65	·75	·71	·58	·46	·65	·57	·54	·57	·63	·62
H.W./S.L.	·54	·62	·69	·63	·51	·42	·65	·58	·55	—	—	·58
$\frac{\text{Leg L.} \times 1000}{\text{C.D.} \times \text{C.W.} \times \text{S.L.}}$	—	—	—	—	—	—	·49	·57	—	—	—	·53
C.C./L.	·38	·47	·58	·63	·61	·53	·50	·47	·10	·45	·43	·47
C.C./S.L.	·15	·32	·50	·59	·55	·48	·41	·54	·13	—	—	·41
S.L./L.	−·22	—	·28	—	—	·19	—	−·06	−·06	—	—	·03

sample.) For the first time, we believe, quantitative measures of body-build have been computed from seriatim data gathered from an approximately constant sample during the first three years of life, and have been compared with ratings of body-build based upon photographs taken on the same days on which the measurements were made. The ratings, as has been explained, were given independently by two

judges (*B* and *D*) for the eleven ages at which photographs were taken. A third judge (*W*) made ratings for months 15, 18 and 24.

For correlation with the several body-build indices the ratings of the judges were pooled. Table XVIII shows W./L.^{2} correlating higher with the pooled photograph ratings, at almost every age, than any of the other indices which were similarly treated. The relative hip width and the relative chest girth ratios (especially the latter) appear as poor measures of body-build when correlated against the criterion furnished by the judges' ratings. An adaptation* of the Wertheimer-Hesketh index (leg length × 1000/C.D. × C.W. × S.L.) was correlated with the photograph ratings for two ages only, 15 and 18 months, and yielded coefficients of ·49 and ·57 respectively. This index affords an example of an index the validity of which suffers because of the relatively low reliability of some of the measures entering into it. Though this objection probably holds for subjects of ages similar to those used in the present investigation, it does not necessarily hold for more mature subjects. This is indicated by the high correlations reported by Wertheimer and Hesketh between their index extremes and the psychotic types with which, if Kretschmer's conclusions were soundly based, asthenic and pyknic body builds should be correlated.

The sharp drops found in the majority of the *r*'s in Table XVIII at the twenty-fourth month are probably to be attributed to a marked shrinkage in the number of cases available for measurement and photographing on that occasion, due to the illness of a number of subjects, and also to the necessity for eliminating the records of several other subjects who proved unusually refractory on their second birthday. If this one patently atypical correlation be omitted from consideration, the average correlation of the W./L.^{2} indices with the judges' ratings, for ten ages between the limits of one month and three years, is ·78. Including the twenty-fourth month, the mean of the *r*'s is ·76.

Table I (p. 34) gives the self-correlations of the W./L.^{2} index, for one-month intervals throughout the first year, for three-month intervals from 12 to 18 months, and for six-month intervals from 18 to 36 months. Five out of the sixteen coefficients are less than ·80, but from the seventh month through the thirty-sixth (excluding the twenty-four month correlations as atypical, for the reasons given above) these self-correlations average ·87. Including all ages, they average ·82. Considering the rapid growth changes, especially in the early months, these correlations are probably minimum measures of reliability; and remeasure-coefficients derived from measurements and remeasurements made on the same day would doubtless be found to be higher.

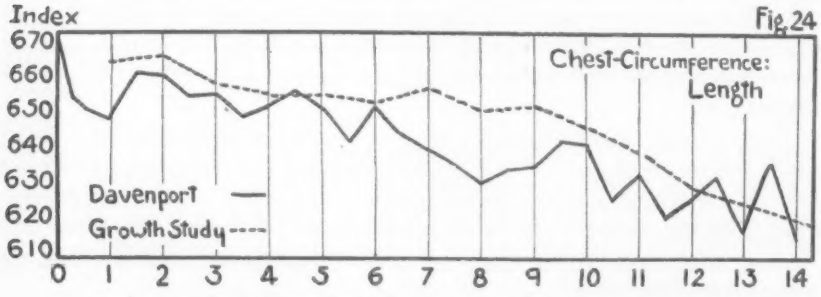
Fig. 22 (p. 59) presents a curve for the development of body-build, using W./L.^{2} as the best quantitative measure of body-build for the ages under consideration, for the sexes separately, and for the total group. Since a low index means relative slenderness and a high index indicates relative stockiness, it is seen that these

* This adaptation was necessary, since we did not make exactly the same measures as did Wertheimer and Hesketh (40).

curves make manifest a trend toward increasing stockiness from the first to the eleventh month, at which time the curves change direction. Two judges (*B* and *W*) studied the entire series of photographs, independently, in an attempt to estimate the age at which these subjects ceased to increase in stockiness and started to grow more slender. *B* made self-comparisons of the photographs of 52 cases and estimated the peak in the trend toward increasing stockiness for each child. The mean of these was 12·3 months. *W* similarly studied the photographs of 50 cases and arrived at a corresponding estimate of 11·6 months. The close agreement between the evidence afforded by the W./L.^{3} curves and these independent estimates from photographs, with respect to the time of occurrence of this change in trend, lends further support to claims for the validity of the W./L.^{3} index as an indicator of body-build in infants and young children.

Before leaving this matter of the curve of development of body-build, however, we should consider some other indices, and the claims made for them. One index proposed by Davenport is that of chest girth, divided by length, or relative chest girth (14). This index presents a picture of the development of body-build different from the W./L.^{3} curve. Davenport concludes (on the strength of the data presented graphically in his Fig. 2, p. 13) that "at six weeks the body has reached its maximum postnatal chubbiness. Another temporary loss in chubbiness occurs at about eight months, due perhaps to the cutting of the incisors." Our W./L.^{3} curve, however, puts the time of maximum chubbiness at eleven months instead of six weeks, and there is no such sudden falling off in chubbiness at eight months as portrayed in Davenport's curve. We incline to the belief that the irregularities in Davenport's curve, including the sharp drop at eight months, are extraneous, and attributable primarily to racial heterogeneity in the cross-sectional samples, to the fewness of the cases in some of his samples (the number of cases in an age group ranges from 9 at month 12 to 37 at month 7), and to the fact that the measurements were taken by different examiners. This is increasingly evident when his curve is compared with our curve of relative chest girth (Fig. 24). Our curve shows an index decreasing with age from the 1st month (especially after the 9th month) which is far more regular than Davenport's in its trend, showing no radical change in plumpness at the 8th month.

Even allowing for the fact of selection in our own sample, and for the relatively



Davenport's Curve of Relative Chest Girth compared with the Growth Study Curve.

small number of cases, it still seems probable that the true curve of development in body-build is more nearly approached with the data from the present sample, and using the W./L.^{2} index, than with Davenport's data, simply because the data are seriatim rather than cross-sectional, the measurements were taken by one person, and a more reliable and valid index of build is used. That our curve is typical is evidenced by the fact that the age curves of the W./L.^{2} indices given in other studies, e.g. Quetelet (34) and Davenport (15), are all very similar, in this early portion of the curve, to our own.

Further comparisons of the indices are afforded by their inter-correlations. These inter-correlations, arranged in order of size, are presented in Table XIX.

TABLE XIX.

Inter-Correlation of Indices of Body-build.

	Months					Mean
	1	5	9	18	36	
H.W./L. × H.W./S.L.	·861	·952	·938	·824	—	·894
C.C./S.L. × C.C./L.	·729	·877	·826	·776	—	·802
H.W./L. × W./L.^{2}	·804	·753	·706	·800	·689	·750
H.W./S.L. × W./L.^{2}	·648	·653	·553	·654	—	·627
C.C./L. × W./L.^{2}	·454	·602	·624	·721	·570	·594
H.W./L. × C.C./L.	·397	·601	·549	·706	·415	·534
H.W./S.L. × C.C./S.L.	·327	·542	·623	·607	—	·525
C.C./S.L. × W./L.^{2}	·218	·503	·455	·624	—	·450
H.W./S.L. × C.C./L.	·204	·538	·482	·462	—	·422
H.W./L. × C.C./S.L.	·157	·468	·503	·498	—	·407
W./L.^{2} × S.L./L.*	·314	·444	—	·272	·141	·272

* Additional correlations between W./L.^{2} and S.L./L. at 12 and 24 months are +·326 and +·136, respectively.

In general, the indices which correlate most highly with each other are those in which there is high correlation (or identity) between the measures included in the two indices. There is also some evidence that the higher correlations are between indices composed of the more reliable measures. As between highly inter-correlated indices, those composed of the most reliable measures are of course more desirable as measures of build. The two highest *r*'s are between measures in which the numerators are identical, and the denominators are length and stem length. Since the former is a more reliable measure, other things being equal, it should be considered the better index. The next three correlations, in order, are comparisons between width-length relations and W./L.^{2}. Since the latter is made up of far more reliable measures than the others, it is obviously a more reliable index, and probably, also, a more complete measure of the width-length relationship than are any of these other indices with which it is compared*.

[* The authors do not seem to have considered the influence of "spurious" correlation on the correlation of these indices. ED.]

Of the other comparisons, we must consider the fact that where there is little correlation between two indices which are composed of reliable measures, either may be a valuable measure of differing body-build traits. In this connection, the correlations between W./L.^{2} and S.L./L., which average only ·27, are worthy of consideration. These two indices are composed of the three most reliable body measures we have taken (not including head measures), yet they show practically no relationship to each other. The former correlates closely with the judges' estimates of chubbiness, while the latter shows no relation to these estimates (Table XVIII, p. 60). It is in no way a measure of lateral versus linear tendency, but of independent linear proportions. It is possible that in considering the relationship of body-build to mental characteristics, both of these indices should be used. The Wertheimer-Hesketh index, which takes account of both relationships, may correlate less with our empirical criterion in part because of this fact. Yet for this very reason, it may be more discriminating (where the individual measurements are reliable) in picking out the extreme builds in which short legs and long thick bodies (pyknic) go together and those in which long legs and short thin bodies (asthenic) coincide. The nature of the correlations between the two separate indices is evidence that for these early years at least, there is no tendency for such body-builds to occur much more frequently than would be expected from a chance distribution in which these are the extreme deviates.

So far in our discussion, we have treated body-build as relative chubbiness or width-length relationship including subcutaneous tissue as well as skeletal structure. If we wish to limit our definition of build to the skeletal structure, we cannot with any certainty draw conclusions from our measures of external body dimensions, since these must always be affected to some extent by subcutaneous tissue. The width measure which is most reliable, hip width, is probably, because it is a simple linear measure, freer from the influence of subcutaneous tissue than is weight, and so may be a better index of skeletal proportions. The H.W./L. index (Fig. 15, p. 58) shows a change in proportions with age which does not coincide with our criterion of ratings of build. According to this index, there is a rapidly increasing lateral tendency through the sixth month, followed by an almost equally rapid decrease after the seventh month. But since the judges' ratings were made from photographs, they were necessarily affected by subcutaneous tissue, and need not be an adequate criterion of skeletal proportions alone. The apparent increasing chubbiness between the seventh and eleventh months might be due entirely to general growth in subcutaneous tissue. Since we have seen in Figs. 9 and 10 that hip width, chest width and shoulder width all continue to grow rapidly throughout the first year, we should expect increasing width relative to length for a longer period than six months. The absolutely greater increase of length, which is a much larger diameter, would mask a tendency to continued increase in chubbiness when the comparison is made with only one of the width diameters, and for this reason it is doubtful if the H.W./L. index gives us a true picture of changing build, either of skeletal proportions, or of proportions which include other tissues in addition to bony structure.

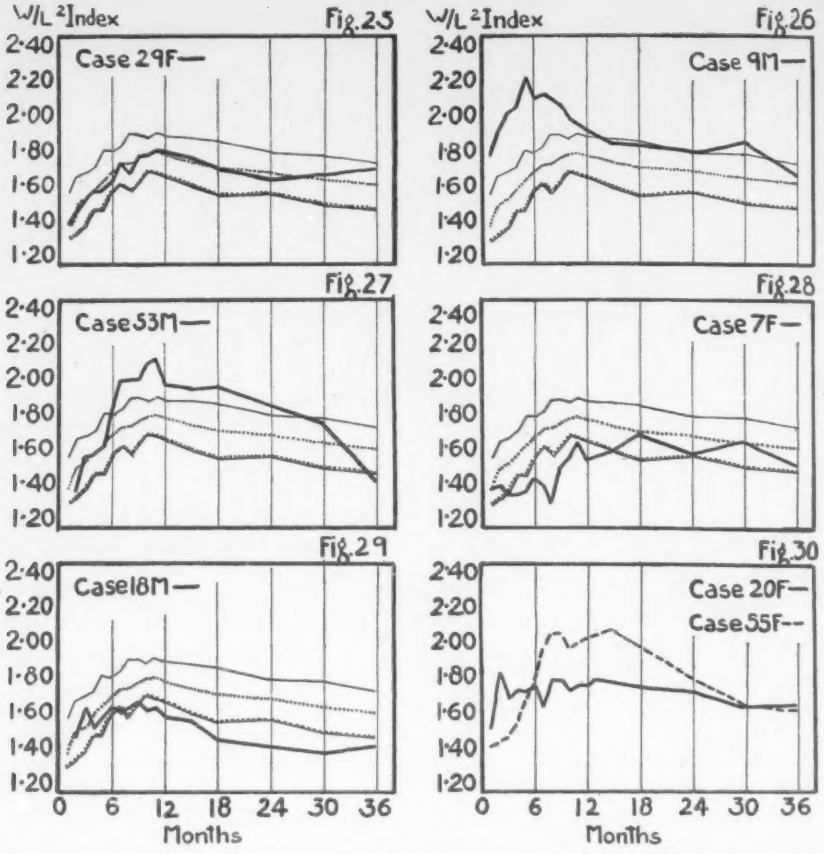
G. CONSISTENCY OF BUILD.

Of the indices of body-build which we have computed, $W./L.^2$ has been found by an empirical criterion, which is based on subcutaneous tissue as well as skeletal proportions, to be the most valid measure of growth changes in chubbiness or slenderness of build. We have also found that it shows a high degree of consistency in consecutive measurements. How consistent is it over longer periods? Granting average changes in body-build, for the total group, do the slender children remain relatively slender, and do the stocky infants continue to be the stocky three-year-olds?

(1) *Individual Curves.*

Figs. 25—29 (p. 66) show growth curves in body-build ($W./L.^2$) for selected subjects. Judge *B*, before consulting the body-build indices, named two cases whom she considered to be of definitely stocky, chubby build, two others whom she classified as definitely slender, and one other whose body-build, in her estimation, approximated the average of the group. The close correspondence between the group's average and the individual body-build curve of case 29 F (the one selected as "typical" by judge *B*) is clearly indicated in the figure. Case 9 M, one of the two subjects estimated by *B* as having a definitely stocky build, is seen to be far above the group average for the first ten months of postnatal life. He reached his peak, however, in the curve of developing stockiness, at the age of five months, as against the group average of eleven months. But although his curve from the tenth month is much closer to the group's average curve of build than was the case in the early months, case 9 M throughout is distinctly atypical in his tendency toward pronounced stockiness. Case 53 M, the other subject selected as definitely above the average in stockiness by judge *B*, presents a very different developmental picture. For the first five months this infant was fairly typical in his weight to height proportions, but between his fifth and seventh month he gained far more rapidly in weight than in length, and more rapidly than did the group as a whole, with the result that his $W./L.^2$ index changed strikingly, as indicated in the figure. The relative rôle of endogenous and exogenous factors in such sudden and unusual changes as this one is a matter which needs investigation. Since the group is a relatively homogeneous one, with the best of dietary advice equally at the command of all of the subjects' parents, the inference would seem to be indicated that such sudden shifts in metabolism are due largely to endogenous factors. When the shift is in the direction of a sudden sharp decrease in weight, rather than an increase, a history of recent illness would of course be looked for before attempting to explain the decrease as due to the influence of postnatal maturational factors. It is of interest to note that case 53 M's curve remains well above the average from the seventh month on, pointing to the establishment of a somewhat stable metabolic equilibrium in his organism at this supernormal level.

Case 7 F presents the picture of a subject who, during the first three years of postnatal life, is consistently more slender than the average. From the third month



Figs. 25—29. Individual Curves of Growth in the Weight : Length² Index compared with Curves of the Mean and Plus and Minus one Standard Deviation.

Fig. 30. Weight : Length² Curves of two Cases who differ widely in their Tendency toward Constant Weight : Length Proportions.

to the twelfth she is strikingly atypical in her tendency toward extreme slenderness. During the ten-months' interval between her eight- and eighteen-month measures she gained in weight at a relatively faster rate than in height, even though after month eleven the group as a whole was showing the contrary tendency, and this deviation from the typical developmental trend results in this subject's close approach to the normal at the eighteenth month. During the next six months, however, her curve changes direction once more and now falls away more sharply than does the average of the group, so that by the twenty-fourth month she is again well below normal. Case 18 M is an individual who, except for the third

month, is also consistently more slender than the normal infant. In contrast to 7 F, however, this atypically slender baby reached his peak in the curve of increasing stockiness at the ninth month, and from that point on deviates more and more sharply from the mean.

(2) *Correlations.*

These individual curves indicate the results which we may expect when the indices are correlated over longer intervals to determine their consistency. The effect of sudden pronounced shifts—like those shown by case 53 M between the fifth and seventh months upon the rank orders of the subjects and consequently upon the correlation coefficients for the intervals of time during the course of which these shifts occurred—is obvious. When individual organisms are going through such rapid and pronounced readjustments, the correlation coefficients for the body characters involved in such readjustments will drop. A study of the consistency correlations, therefore, should be informative with respect to which developmental periods tend to be characterised by these pronounced individual divergences from the mean curve of growth, and which ones manifest relatively stable developmental characteristics on the part of the individual members of the group. Table XX gives these data.

Self-Correlations showing Consistency of Body-Build.

TABLE XX.

$W./L.^2$

Month	3	6	12	18	24	30	36
1	·68	·48	—	—	·17	—	·10
3	—	·64	—	·17	·31	—	·28
6	—	—	·44	—	·24	—	·28
12	—	—	—	·79	·74	—	·75
18	—	—	—	—	·73	—	·80
24	—	—	—	—	—	·74	·73
30	—	—	—	—	—	—	·91

It is evident at a glance that something is operating to reduce the longer-interval correlations into which the first month measurements enter. The $W./L.^2$ correlations, while they are ·718 for the first and second months, and ·741 for the second and third months (see Table I), become ·68 for the first and third months (Table XX) and drop to ·48 and to ·17 for the first and sixth months and the first and twenty-fourth months, respectively. Similarly, the correlations of the sixth month indices with those for the twelfth, twenty-fourth and thirty-sixth months give the following coefficients, ·44, ·24 and ·28. But when the twelve-month indices are correlated with those for the eighteenth, twenty-fourth and thirty-sixth months we find a considerably higher degree of stability in the rank orders of the subjects at these ages, with coefficients of ·79, ·74 and ·75. Over the eighteen-month interval between the ages of 18 and 36 months, our subjects maintain their relative positions with

respect to their W./L.^{2} index to the extent indicated by a coefficient of correlation of the order of ·80, while for the thirtieth and thirty-sixth months the correlation is ·91.

The consistency correlations for the H.W./L. index (Table XXI) are very similar to those for W./L.^{2}. The *r*'s of the former are more often larger when the first and sixth months are correlated with later ages, while the latter are higher when the twelfth and twenty-fourth months are compared with later ages. None of the differences, however, is great enough to be significant. And, in general, the H.W./L. index shows no greater consistency in build than does the W./L.^{2} index.

These correlations, and also the individual curves, show that there is little consistency in lateral-linear build during the first year—just as there is little consistency in the relative size of weight and all width measures. But after the twelfth month the correlations increase somewhat and the children appear to have established a body-build which is comparatively stable—at least over a two-year period.

Self-Correlations showing Consistency of Body-Build.

TABLE XXI.

H.W./L.

Month	6	12	24	36
1	·50	·24	·27	·13
6	—	·34	·47	·46
12	—	—	·70	·66
24	—	—	—	·72

TABLE XXII.

S.L./L.

Month	5	12	18	24	36
1	+·12	−·05	—	−·11	−·07
5	—	+·30	—	+·26	+·16
12	—	—	+·28	+·37	+·13
18	—	—	—	—	+·62
24	—	—	—	—	+·42

Consistency correlations were computed for another index composed of reliable measures, S.L./L. (Table XXII). All the coefficients are low for all ages, even though the measures used in this index are relatively consistent. It is probable that since the legs are continuing to grow rapidly through the thirty-sixth month, individual differences in growth trends would obliterate any tendency for consistent stem-stature relationships within the age limits here considered*.

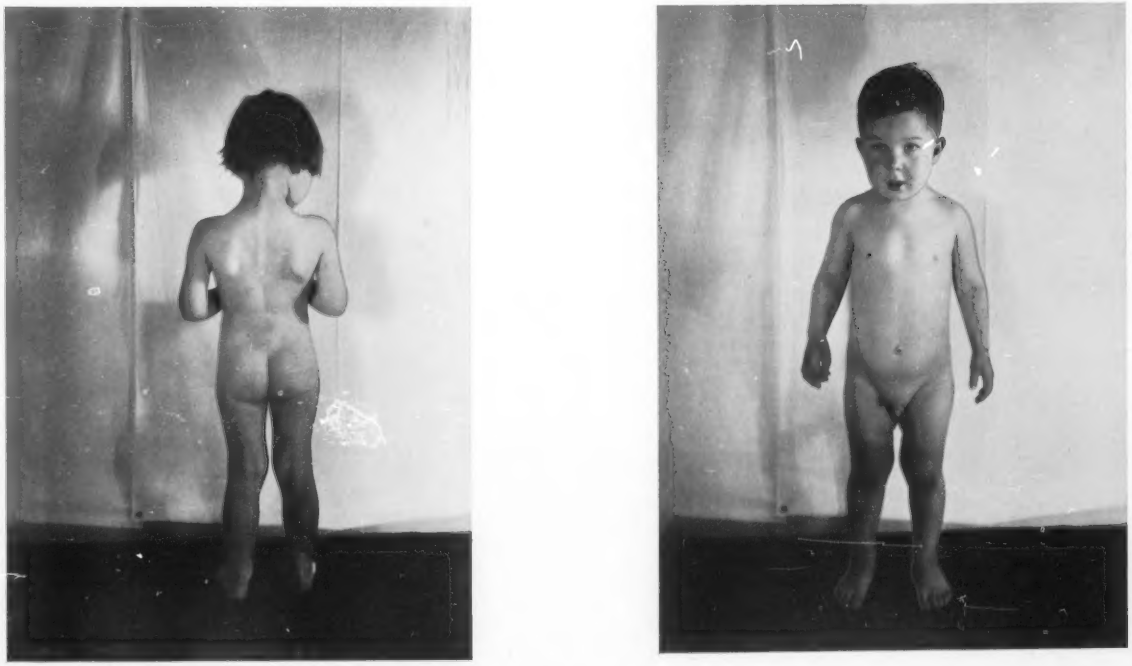
Body-build types may be manifested during the first year in the growth *changes* of the W./L.^{2} index rather than in the child's relative build at any given age. As we have seen (Figs. 25—29, p. 66), when the age curves of this index are plotted for each child separately, we find wide individual difference in the trends of the curves. Some of the children (for example, cases 9 M and 18 M) show an early

* Hejinian and Hatt (25) plotted individual curves of the percentile rank of S.L./L. index for 50 children aged 24 to 60 months. Seventy-two per cent. of these showed consistency of build, but the others distinctly changed their percentile rank. This body proportion, then, remains unstable for many children even to five years of age.

Plate I (a). 1 month

Plate I (b). 5 months

Plate I (c). 30 months



Comparisons between a slender child, 7 *F* (left), and a stocky child, 9 *M* (right), at three age levels: 1, 5 and 30 months. 9 *M* was extremely chubby at 5 months.

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Plate II (a). 1 month

Small child

Large child

Plate II (b). 9 months

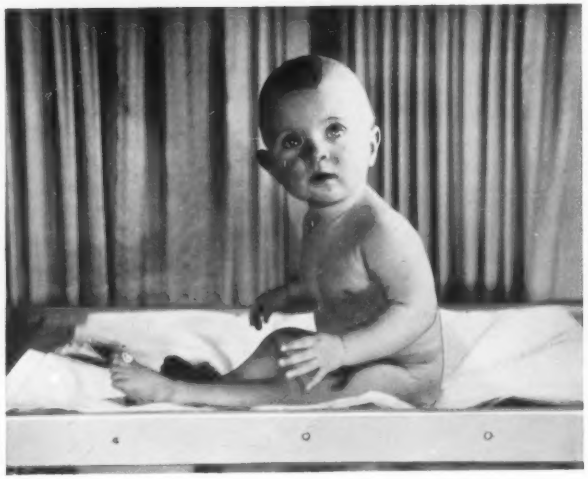

Plate II (c). 36 months

Comparisons between a consistently small child (left) and a consistently large child (right) at three age levels: 1, 9 and 36 months.

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increase in chubbiness, and others a late increase (case 7 F), as compared with the majority, who follow the curve of the mean. A number of the more slender children have little or no tendency toward increasing W./L.^{2} indices during the first year, but maintain a comparatively constant index throughout the entire three years. The W./L.^{2} curves of two girls, cases 20 F and 55 F (Fig. 30), illustrate the wide divergence from the mean in tendency to change in body-build. Case 20 has an index which is close to the mean for the first four months. After this time it increases rapidly to the eighth month, stays high through the fifteenth month, and then decreases at a rate far more rapid than the average, until at the thirtieth and thirty-sixth months her index has returned very near to the mean. Case 55, on the other hand, has indices which are seldom far from the mean, though they are well above it during the first six months. Her index curve is almost a straight line, and, except for the rise between the first and second months, and a slow decline after the fifteenth month, shows no tendency towards a changing W./L.^{2} index.

When the group is divided into halves, one half containing children with less than average tendency to increased W./L.^{2} index during the first year, and the other half containing those with greater tendencies to chubbiness, the slender group, as would be expected, has a much higher consistency correlation between the sixth and twelfth months (r is ·87 for 25 cases) than is found for the total group (·44). However, their consistency correlations drop later, and are lower than for the total in the r between measures at the twelfth and twenty-fourth months (·53) and between the twenty-fourth and thirty-sixth (·57). Of the entire group of children, only one (46 M, Plate I, see p. 32) was found to be so consistently heavy that his W./L.^{2} index did not once fall within one standard deviation of the mean for the 17 ages at which the children were measured. One other child (9 M, Plate II, see p. 32) remained above one standard deviation for all but three ages. Of the slender children, one (57 F, Plate I) remained below one standard deviation for all but two ages, while another (27 F, Plate II) remained below for all but three ages.

Infant trends in body build may be characteristic of glandular and metabolic balances in the individual and so predictive of build later in life. This we cannot know until we can compare these early trends with measures of the same individuals later. But we have found that no single index (composed of the measures we have taken) of an infant under one year can be considered as representative of a constitutional type which will continue to be one of its characteristics for any protracted length of time.

SUMMARY AND CONCLUSIONS.

1. A seriatim growth study of nine measures of body size was made on a group of from forty-six to sixty-one infants who were measured at 17 ages during their first 36 months.

2. The measures were found to be sufficiently reliable to indicate growth trends of individuals; there are indications that some measures are more reliable than

others, the highest remeasurement-coefficients being found for length, weight and stem length.

3. The group was found to be typical of other groups reported in the literature, except that after the sixth month its members averaged taller and heavier than children in other parts of the country, with the exception of selected groups such as those mentioned in the recent Iowa studies.

4. Growth in all dimensions was retarded, rapidly at first, and then more slowly through 36 months.

5. In proportion to their size at the first month, the width measures of hip, chest and shoulder increased more rapidly during the first year than any other measure except weight. After this age, the greatest proportionate growth was in length. Chest depth increased even less than did head circumference.

6. In variability there were no significant changes with growth, nor between the sexes.

7. The boys were consistently a little larger for all measures than the girls.

8. Correlations over the three-year period show that head circumference, length and stem length are the most consistent measures; that is, children with large heads at the first month tend to have relatively large heads at the thirty-sixth month and those with small heads usually continue to have small heads, while the same is true of lengths. Weight and width measures show only slight consistency before the twelfth month, but with the exception of two unreliable measures (chest width and depth) are fairly consistent after this age.

9. Hip width and weight are more closely correlated than any other pair of measures. Length and stem length are also highly correlated. Other measures of body width correlate fairly highly with weight; while the width-length comparisons show less relationship though they are still positively correlated. The smallest correlations are found between shoulder width and stem length and average ·53.

10. There is a decrease in some of the correlations between measures of widths and lengths during the first half of the first year. This is true of the *r*'s between weight and length, hip width and length, hip width and stem length, and less clearly between hip width and shoulder width. These decreased *r*'s appear to occur more often at ages when the measures involved are growing at disproportionate rates. Thus, the width-length correlations drop during the ages when most children are rapidly becoming chubbier.

11. Growth changes in body proportions are shown by curves of indices, comparing length measures with width measures and with weight. Most of these show proportionately increasing width during some portion of the first year.

12. A comparison of these indices with ratings of body build leads us to the conclusion that $W./L.^3$ is (of the indices studied) the most valid measure of relative chubbiness, or lateral-linear tendencies in build.

13. The W./L.^{2} index shows a curve of rapidly increasing chubbiness during the first eleven months, with a slow decrease in chubbiness after this age.

14. Of the three indices of build whose values were correlated for consistency over a long age interval, namely, W./L.^{2}, H.W./L. and S.L./L., none indicated consistency of build during the first year, though W./L.^{2} and H.W./L. show increasing consistency after this age.

15. We cannot predict from indices at an early age what an individual's body-build will be later. It may be possible, however, to predict later build from early trends in build. Our data give no clear evidence on this point.

16. No significant sex differences in rates of growth were found for this group, though the boys were found to be slightly but consistently heavier for their length than the girls.

NOTE. Since this memoir was set up a letter has reached the Editor from Dr Bayley, which he thinks somewhat explains their different conceptions of "chubbiness." No relation of width to length would conform to his conception of chubbiness = plumpness. Nor did he after a study of Kretschmer's work based on no extensive statistical investigation give credit to his classifications or conclusions. The following are the important paragraphs from Dr Bayley's letter:

"I have been trying out some of the indices you suggested. They should prove of value in studies of nutrition. However, in the study under consideration, I was most interested in finding an index which would differentiate types of body-build such as those described by Kretschmer (pyknic-leptosome), or Davenport (lateral-linear), and not so much interested in obtaining a measure of nutrition or health. This latter problem is one I intend to take up in connection with a study of physicians' ratings of the children's health.

"It would, of course, be of value to find an index which differentiates the children and is constant at all ages, and in which the individual children maintain the same relative position in the group. None of the indices for which I have made the comparisons has remained consistent, so that no child who is classified in body-build at one age can be expected to have the same build at another age. "Chubbiness" as we used it in our ratings of photographs meant the tendency to be wide, relative to length, rather than to have excess fatty tissue, though this latter factor had a strong influence in the rating.

"It still seems to me that the index I am looking for is one which expresses width relative to length, and all of my data indicate that such an index would show changes with age, and would not be constant. This finding in itself should be of importance in understanding the processes of growth in young children."

TABLE XXIII.

ANTHROPOMETRIC MEASURES—INFANT GROWTH STUDY.

(*a*) *Chest Circumference Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
40·0	1	43·0	1	44·0	2	46·5	1	47·5	2	49·5	1
39·5	2	42·5	—	43·5	—	46·0	—	47·0	2	49·0	1
39·0	—	42·0	2	43·0	1	45·5	1	46·5	—	48·5	—
38·5	—	41·5	2	42·5	1	45·0	1	46·0	1	48·0	1
38·0	2	41·0	—	42·0	7	44·5	—	45·5	2	47·5	—
37·5	1	40·5	1	41·5	4	44·0	1	45·0	2	47·0	—
37·0	6	40·0	3	41·0	6	43·5	2	44·5	5	46·5	5
36·5	2	39·5	5	40·5	1	43·0	5	44·0	3	46·0	4
36·0	8	39·0	8	40·0	3	42·5	4	43·5	5	45·5	1
35·5	6	38·5	4	39·5	4	42·0	6	43·0	5	45·0	5
35·0	7	38·0	7	39·0	8	41·5	4	42·5	3	44·5	4
34·5	3	37·5	3	38·5	6	41·0	4	42·0	6	44·0	6
34·0	2	37·0	3	38·0	1	40·5	5	41·5	4	43·5	2
33·5	1	36·5	2	37·5	—	40·0	5	41·0	3	43·0	8
33·0	3	36·0	4	37·0	4	39·5	3	40·5	3	42·5	4
32·5	—	35·5	4	36·5	2	39·0	4	40·0	2	42·0	3
32·0	1	35·0	1			38·5	1	39·5	1	41·5	1
						38·0	2	39·0	—	41·0	1
						37·5	—	38·5	1	40·5	2
						37·0	—			40·0	—
										39·5	1

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
51·0	2	50·5	—	51·0	—	51·0	4	52·0	1	52·5	1
50·5	—	50·0	3	50·5	2	50·5	1	51·5	2	52·0	—
50·0	—	49·5	—	50·0	2	50·0	1	51·0	—	51·5	3
49·5	—	49·0	1	49·5	3	49·5	3	50·5	—	51·0	1
49·0	1	48·5	3	49·0	2	49·0	1	50·0	1	50·5	—
48·5	1	48·0	1	48·5	2	48·5	5	49·5	5	50·0	1
48·0	2	47·5	3	48·0	2	48·0	4	49·0	2	49·5	1
47·5	4	47·0	4	47·5	5	47·5	4	48·5	—	49·0	3
47·0	2	46·5	8	47·0	5	47·0	5	48·0	1	48·5	2
46·5	3	46·0	2	46·5	5	46·5	5	47·5	6	48·0	2
46·0	6	45·5	6	46·0	2	46·0	3	47·0	3	47·5	3
45·5	2	45·0	1	45·5	6	45·5	4	46·5	3	47·0	3
45·0	6	44·5	3	45·0	5	45·0	1	46·0	5	46·5	2
44·5	1	44·0	4	44·5	2	44·5	2	45·5	1	46·0	5
44·0	5	43·5	4	44·0	1	44·0	3	45·0	2	45·5	3
43·5	2	43·0	2	43·5	—	43·5	—	44·5	1	45·0	3
43·0	3	42·5	—	43·0	1	43·0	—	44·0	1	44·5	1
42·5	3	42·0	1	42·5	—	42·5	2	43·5	—	44·0	—
42·0	4			42·0	—			43·0	—	43·5	1
				41·5	1			42·5	—		
								42·0	1		

TABLE XXIII—(*continued*).

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
53·0	1	54·5	1	55	1	55	1	56·5	1
52·5	1	54·0	—	54	2	54	4	56·0	—
52·0	1	53·5	—	53	2	53	1	55·5	2
51·5	2	53·0	2	52	5	52	5	55·0	—
51·0	—	52·5	2	51	7	51	7	54·5	4
50·5	1	52·0	2	50	8	50	7	54·0	1
50·0	7	51·5	—	49	5	49	5	53·5	1
49·5	2	51·0	1	48	2	48	2	53·0	2
49·0	2	50·5	2	47	1	47	2	52·5	3
48·5	2	50·0	6	46	3	46	2	52·0	2
48·0	4	49·5	3	45	1	45	1	51·5	2
47·5	2	49·0	1					51·0	7
47·0	2	48·5	2					50·5	5
46·5	6	48·0	4					50·0	5
46·0	2	47·5	4					49·5	2
45·5	—	47·0	4					49·0	4
45·0	2	46·5	—					48·5	1
44·5	—	46·0	2					48·0	1
44·0	1	45·5	2					47·5	2
								47·0	3
								46·5	1

(*b*) *Hip Width Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
12·9	2	14·3	2	14·9	2	15·4	1	16·2	1	17·4	1
12·6	1	14·0	—	14·6	1	15·1	—	15·9	2	17·1	—
12·3	1	13·7	3	14·3	2	14·8	4	15·6	1	16·8	—
12·0	—	13·4	1	14·0	3	14·5	8	15·3	4	16·5	1
11·7	5	13·1	1	13·7	6	14·2	5	15·0	7	16·2	3
11·4	6	12·8	3	13·4	6	13·9	8	14·7	4	15·9	1
11·1	2	12·5	3	13·1	9	13·6	3	14·4	3	15·6	5
10·8	8	12·2	8	12·8	5	13·3	8	14·1	9	15·3	4
10·5	6	11·9	6	12·5	6	13·0	10	13·8	2	15·0	3
10·2	5	11·6	2	12·2	3	12·7	3	13·5	7	14·7	8
9·9	7	11·3	8	11·9	2	12·4	3	13·2	4	14·4	10
9·6	2	11·0	7	11·6	4	12·1	—	12·9	3	14·1	2
9·3	2	10·7	2	11·3	1	11·8	1	12·6	4	13·8	7
9·0	2	10·4	—	11·0	3	11·5	3	12·3	2	13·5	1
		10·1	2	10·7	2	11·2	—	12·0	1	13·2	4
		9·8	1	10·4	—	10·9	1	11·7	1	12·9	4
				10·1	1	10·6	—			12·6	1
						10·3	1				

TABLE XXIII—(*continued*).

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
18·0	1	18·0	1	18·3	1	17·9	1	17·8	2	18·1	2
17·5	1	17·7	—	18·0	—	17·6	—	17·5	1	17·8	1
17·0	—	17·4	1	17·7	—	17·3	2	17·2	1	17·5	1
16·5	3	17·1	3	17·4	—	17·0	4	16·9	2	17·2	2
16·0	2	16·8	2	17·1	1	16·7	1	16·6	1	16·9	—
15·5	9	16·5	1	16·8	4	16·4	4	16·3	5	16·6	6
15·0	7	16·2	7	16·5	2	16·1	2	16·0	4	16·3	4
14·5	12	15·9	1	16·2	2	15·8	4	15·7	3	16·0	3
14·0	5	15·6	3	15·9	3	15·5	5	15·4	2	15·7	5
13·5	7	15·3	6	15·6	9	15·2	5	15·1	2	15·4	3
13·0	2	15·0	5	15·3	2	14·9	5	14·8	4	15·1	1
12·5	2	14·7	7	15·0	7	14·6	4	14·5	2	14·8	1
		14·4	4	14·7	3	14·3	—	14·2	2	14·5	2
		14·1	—	14·4	5	14·0	4	13·9	3	14·2	2
		13·8	6	14·1	4	13·7	2	13·6	1	13·9	—
				13·8	2	13·4	4			13·6	2
				13·5	1						
				13·2	1						
				12·9	—						

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
18·3	2	18·9	—	19·7	1	19·6	—	19·9	1
18·0	—	18·6	3	19·4	—	19·3	1	19·6	4
17·7	1	18·3	—	19·1	—	19·0	1	19·3	1
17·4	3	18·0	3	18·8	—	18·7	2	19·0	2
17·1	2	17·7	2	18·5	2	18·4	2	18·7	4
16·8	1	17·4	2	18·2	4	18·1	5	18·4	4
16·5	5	17·1	3	17·9	1	17·8	5	18·1	13
16·2	3	16·8	2	17·6	5	17·5	6	17·8	8
15·9	2	16·5	5	17·3	7	17·2	5	17·5	1
15·6	4	16·2	4	17·0	4	16·9	4	17·2	2
15·3	4	15·9	6	16·7	3	16·6	2	16·9	4
15·0	5	15·6	1	16·4	5	16·3	1	16·6	2
14·7	2	15·3	3	16·1	1	16·0	1	16·3	1
14·4	1	15·0	1	15·8	—	15·7	—	16·0	—
14·1	—	14·7	—	15·5	—	15·4	—	15·7	1
13·8	—	14·4	—	15·2	—	15·1	—	15·4	1
13·5	—	14·1	—	14·9	3	14·8	1		

TABLE XXIII—(*continued*). (*c*) *Chest Depth Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
11·0	1	12·6	1	12·4	—	12·4	—	12·4	1	13·2	1
10·8	1	12·3	—	12·2	1	12·2	—	12·3	—	13·0	—
10·6	2	12·0	2	12·0	—	12·0	2	12·2	—	12·8	—
10·4	5	11·7	—	11·8	1	11·8	2	12·1	—	12·6	—
10·2	5	11·4	3	11·6	1	11·6	2	12·0	3	12·4	—
10·0	6	11·1	4	11·4	1	11·4	4	11·9	—	12·2	—
9·8	6	10·8	7	11·2	5	11·2	4	11·8	—	12·0	4
9·6	5	10·5	8	11·0	2	11·0	2	11·7	1	11·8	1
9·4	4	10·2	14	10·8	6	10·8	5	11·6	1	11·6	1
9·2	5	9·9	6	10·6	5	10·6	5	11·5	—	11·4	5
9·0	3	9·6	8	10·4	8	10·4	8	11·4	2	11·2	11
8·8	1	9·3	3	10·2	6	10·2	6	11·3	6	11·0	1
8·6	3	9·0	1	10·0	1	10·0	6	11·2	4	10·8	10
				9·8	6	9·8	2	11·1	1	10·6	5
				9·6	4	9·6	3	11·0	7	10·4	2
				9·4	3	9·4	1	10·9	4	10·2	3
				9·2	2	9·2	1	10·8	1	10·0	1
				9·0	1			10·7	2	9·8	—
								10·6	4	9·6	1
								10·5	2	9·4	1
								10·4	6		
								10·3	4		
								10·2	1		
								10·1	—		
								10·0	1		
								9·9	—		
								9·8	1		
								9·7	1		
								9·6	1		

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
12·6	1	13·4	1	13·8	—	13·7	1	13·2	1	13·9	—
12·5	2	13·2	1	13·6	—	13·5	—	13·0	—	13·6	1
12·4	—	13·0	1	13·4	—	13·3	—	12·8	2	13·3	1
12·3	—	12·8	—	13·2	—	13·1	—	12·6	—	13·0	—
12·2	2	12·6	—	13·0	1	12·9	3	12·4	2	12·7	3
12·1	2	12·4	—	12·8	1	12·7	1	12·2	12	12·4	4
12·0	1	12·2	1	12·6	—	12·5	2	12·1	6	12·1	12
11·9	2	12·0	2	12·4	—	12·3	3	11·9	1	11·8	10
11·8	1	11·8	4	12·2	3	12·1	5	11·7	6	11·5	8
11·7	1	11·6	5	12·0	4	11·9	10	11·5	2	11·2	5
11·6	2	11·4	6	11·8	7	11·7	3	11·3	4	10·9	2
11·5	2	11·2	9	11·6	8	11·5	6	11·1	4	10·6	—
11·4	3	11·0	6	11·4	7	11·3	7	10·9	2	10·3	1
11·3	1	10·8	2	11·2	2	11·1	2	10·7	2	10·0	—
11·2	4	10·6	2	11·0	3	10·9	4				
11·1	3	10·4	2	10·8	3	10·7	—				
11·0	2	10·2	1	10·6	3	10·5	—				
10·9	1	10·0	1	10·4	1	10·3	1				
10·8	4			10·2	1						
10·7	1			10·0	—						
10·6	5										
10·5	2										
10·4	2										
10·3	2										
10·2	1										
10·1	1										
10·0	1										
9·9	—										
9·8	—										
9·7	1										

TABLE XXIII—(*continued*).

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
13·3	2	14·8	1	14·6	1	15·8	1	14·6	1
13·1	1	14·6	—	14·4	1	15·5	—	14·4	1
12·9	4	14·4	—	14·2	—	15·2	—	14·2	1
12·7	5	14·2	—	14·0	—	14·9	—	14·0	2
12·5	3	14·0	—	13·8	—	14·6	—	13·8	1
12·3	5	13·8	—	13·6	2	14·3	—	13·6	2
12·1	5	13·6	1	13·4	1	14·0	2	13·4	2
11·9	7	13·4	1	13·2	3	13·7	2	13·2	—
11·7	2	13·2	3	13·0	5	13·4	2	13·0	5
11·5	4	13·0	2	12·8	1	13·1	2	12·8	6
11·3	4	12·8	7	12·6	5	12·8	2	12·6	6
11·1	—	12·6	5	12·4	4	12·5	7	12·4	3
10·9	1	12·4	3	12·2	4	12·2	8	12·2	5
10·7	—	12·2	7	12·0	6	11·9	5	12·0	3
10·5	1	12·0	4	11·8	1	11·6	3	11·8	5
		11·8	3	11·6	3	11·3	3	11·6	2
		11·6	3	11·4	3	11·0	2	11·4	—
		11·4	4	11·2	—	10·7	1	11·2	1
		11·2	—	11·0	1	10·4	2	11·0	1
		11·0	—	10·8	—				
		10·8	1	10·6	1				

(*d*) *Chest Width Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
11·6	4	12·8	1	14·8	—	13·7	2	15·7	1	15·8	1
11·4	2	12·5	—	14·5	1	13·5	5	15·4	—	15·5	—
11·2	2	12·2	3	14·2	1	13·3	2	15·1	—	15·2	—
11·0	3	11·9	4	13·9	1	13·1	1	14·8	—	14·9	1
10·8	2	11·6	3	13·6	—	12·9	1	14·5	—	14·6	2
10·6	2	11·3	8	13·3	1	12·7	2	14·2	1	14·3	1
10·4	4	11·0	5	13·0	4	12·5	4	13·9	3	14·0	4
10·2	3	10·7	7	12·7	4	12·3	6	13·6	4	13·7	1
10·0	3	10·4	4	12·4	2	12·1	5	13·3	4	13·4	3
9·8	8	10·1	2	12·1	5	11·9	2	13·0	7	13·1	10
9·6	6	9·8	6	11·8	6	11·7	4	12·7	8	12·8	8
9·4	2	9·5	1	11·5	8	11·5	5	12·4	10	12·5	1
9·2	3	9·2	1	11·2	8	11·3	7	12·1	3	12·2	6
9·0	1			10·9	7	11·1	6	11·8	7	11·9	2
				10·6	3	10·9	4	11·5	5	11·6	1
				10·3	2	10·7	1	11·2	1	11·3	—
				10·0	3	10·5	—	10·9	—	11·0	2
				9·7	1	10·3	2			10·7	1

TABLE XXIII—(*continued*).

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
15·4	2	17·2	—	16·9	—	16·3	1	16·6	—	17·0	2
15·1	2	16·9	—	16·6	—	16·0	—	16·3	1	16·7	—
14·8	1	16·6	—	16·3	1	15·7	2	16·0	1	16·4	2
14·5	2	16·3	—	16·0	1	15·4	4	15·7	4	16·1	2
14·2	6	16·0	1	15·7	1	15·1	7	15·4	3	15·8	6
13·9	4	15·7	1	15·4	2	14·8	4	15·1	9	15·5	4
13·6	10	15·4	3	15·1	6	14·5	6	14·8	6	15·2	6
13·3	7	15·1	1	14·8	2	14·2	7	14·5	4	14·9	8
13·0	4	14·8	3	14·5	7	13·9	4	14·2	1	14·6	6
12·7	1	14·5	4	14·2	8	13·6	3	13·9	1	14·3	8
12·4	5	14·2	12	13·9	4	13·3	2	13·6	2	14·0	—
12·1	1	13·9	4	13·6	7	13·0	1	13·3	3	13·7	4
		13·6	6	13·3	3	12·7	2	13·0	—	13·4	3
		13·3	5	13·0	2	12·4	—	12·7	1	13·1	1
		13·0	2	12·7	—	12·1	—				
		12·7	2	12·4	3	11·8	—				
		12·4	—	12·1	1	11·5	—				
		12·1	2	11·8	—	11·2	1				
		11·8	2								
		11·5	—								

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
18·4	—	17·3	3	17·9	3	17·9	5	19·1	1
18·1	—	17·0	2	17·6	—	17·6	—	18·8	—
17·8	—	16·7	2	17·3	3	17·3	3	18·5	1
17·5	—	16·4	3	17·0	2	17·0	5	18·2	2
17·2	2	16·1	8	16·7	4	16·7	8	17·9	2
16·9	4	15·8	5	16·4	8	16·4	4	17·6	2
16·6	2	15·5	3	16·1	4	16·1	7	17·3	3
16·3	2	15·2	3	15·8	5	15·8	3	17·0	9
16·0	4	14·9	7	15·5	4	15·5	4	16·7	8
15·7	6	14·6	5	15·2	2	15·2	1	16·4	3
15·4	4	14·3	2	14·9	4	14·9	2	16·1	9
15·1	8			14·6	—	14·6	—	15·8	3
14·8	9			14·3	—	14·3	—	15·5	1
14·5	2			14·0	2	14·0	—	15·2	2
14·2	2					13·7	—		
13·9	—								
13·6	1								
13·3	—								

TABLE XXIII—(*continued*).

(*e*) *Head Circumference Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
41·0	1	43·0	1	44·5	1	45·0	1	46·0	1	47·0	1
40·5	1	42·5	—	44·0	—	44·5	—	45·5	1	46·5	—
40·0	1	42·0	—	43·5	—	44·0	4	45·0	3	46·0	5
39·5	2	41·5	1	43·0	2	43·5	3	44·5	2	45·5	3
39·0	4	41·0	7	42·5	4	43·0	4	44·0	11	45·0	11
38·5	5	40·5	4	42·0	3	42·5	10	43·5	9	44·5	6
38·0	9	40·0	9	41·5	8	42·0	12	43·0	8	44·0	6
37·5	7	39·5	10	41·0	12	41·5	7	42·5	5	43·5	8
37·0	9	39·0	10	40·5	10	41·0	8	42·0	7	43·0	5
36·5	4	38·5	5	40·0	7	40·5	4	41·5	3	42·5	4
36·0	1	38·0	5	39·5	3	40·0	2	41·0	4	42·0	3
35·5	4	37·5	2	39·0	4	39·5	1	40·5	—	41·5	2
35·0	—	37·0	2	38·5	2	39·0	1	40·0	1	41·0	—
34·5	—	36·5	1	38·0	1	38·5	—	39·5	—	40·5	—
34·0	—	36·0	—	37·5	—	38·0	—	39·0	1	40·0	1
33·5	1	35·5	1	37·0	1	37·5	1				

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
48·5	1	48·5	1	49·5	1	50·0	1	50·0	2	51·0	1
48·0	—	48·0	1	49·0	—	49·5	1	49·5	—	50·5	1
47·5	—	47·5	3	48·5	3	49·0	1	49·0	3	50·0	1
47·0	3	47·0	5	48·0	—	48·5	2	48·5	6	49·5	2
46·5	3	46·5	5	47·5	7	48·0	6	48·0	6	49·0	3
46·0	7	46·0	4	47·0	4	47·5	7	47·5	6	48·5	7
45·5	8	45·5	10	46·5	7	47·0	5	47·0	3	48·0	5
45·0	7	45·0	7	46·0	5	46·5	4	46·5	8	47·5	6
44·5	3	44·5	7	45·5	9	46·0	6	46·0	2	47·0	5
44·0	8	44·0	1	45·0	3	45·5	5	45·5	0	46·5	3
43·5	2	43·5	1	44·5	6	45·0	3	45·0	3	46·0	6
43·0	1	43·0	3	44·0	—	44·5	4	44·5	2	45·5	—
42·5	5	42·5	—	43·5	3	44·0	1	44·0	0	45·0	1
42·0	1	42·0	—	43·0	—	43·5	—	43·5	0	43·5	—
41·5	—	41·5	1	42·5	—	43·0	—	43·0	1	43·0	—
41·0	1			42·0	—	42·5	—			42·5	1
				41·5	1	42·0	1				

TABLE XXIII—(*continued*).

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
51·5	2	51·5	4	53·0	1	53·0	1	53·5	1
51·0	2	51·0	2	52·5	1	52·5	2	53·0	1
50·5	1	50·5	2	52·0	3	52·0	2	52·5	2
50·0	4	50·0	4	51·5	3	51·5	5	52·0	1
49·5	3	49·5	2	51·0	1	51·0	3	51·5	4
49·0	9	49·0	6	50·5	2	50·5	4	51·0	9
48·5	4	48·5	10	50·0	4	50·0	6	50·5	4
48·0	8	48·0	1	49·5	3	49·5	8	50·0	8
47·5	5	47·5	8	49·0	8	49·0	1	49·5	2
47·0	5	47·0	1	48·5	4	48·5	5	49·0	4
46·5	4	46·5	3	48·0	1	48·0	2	48·5	4
46·0	1	46·0	—	47·5	4	47·5	2	48·0	—
45·5	2	45·5	1	47·0	—	47·0	—	47·5	1
45·0	—	45·0	—			46·5	—	47·0	—
44·5	—	44·5	1			46·0	1	46·5	1
44·0	1								

(*f*) *Length Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
58·0	1	62·0	1	65·0	1	67·0	2	70·5	1	72·5	1
57·5	2	61·5	2	64·5	2	66·5	2	70·0	1	72·0	1
57·0	2	61·0	2	64·0	1	66·0	3	69·5	1	71·5	1
56·5	5	60·5	4	63·5	3	65·5	4	69·0	1	71·0	2
56·0	6	60·0	4	63·0	5	65·0	6	68·5	4	70·5	3
55·5	4	59·5	2	62·5	1	64·5	1	68·0	2	70·0	2
55·0	3	59·0	4	62·0	3	64·0	6	67·5	2	69·5	3
54·5	3	58·5	4	61·5	4	63·5	3	67·0	7	69·0	2
54·0	3	58·0	9	61·0	10	63·0	7	66·5	2	68·5	3
53·5	3	57·5	5	60·5	4	62·5	4	66·0	3	68·0	4
53·0	3	57·0	6	60·0	5	62·0	4	65·5	6	67·5	7
52·5	6	56·5	—	59·5	6	61·5	3	65·0	4	67·0	7
52·0	3	56·0	6	59·0	2	61·0	6	64·5	5	66·5	2
51·5	4	55·5	2	58·5	2	60·5	2	64·0	3	66·0	4
51·0	3	55·0	—	58·0	1	60·0	3	63·5	5	65·5	3
50·5	—	54·5	3	57·5	3	59·5	1	63·0	4	65·0	4
50·0	1	54·0	2	57·0	2	59·0	1	62·5	1	64·5	1
49·5	2	53·5	2	56·5	2	58·5	1	62·0	1	64·0	4
		53·0	1	56·0	3			61·5	—	63·5	—
		52·5	1					61·0	1	63·0	1
		52·0	1					60·5	—	62·5	1
								60·0	1	62·0	—
								59·5	—	61·5	—
								59·0	—		
								58·5	1		

TABLE XXIII—(*continued*).

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
74·0	1	76·0	1	78·0	1	78·5	2	81·5	1	81·5	2
73·5	1	75·5	3	77·5	—	77·5	2	80·5	1	80·5	3
73·0	3	75·0	—	77·0	—	76·5	3	79·5	1	79·5	—
72·5	1	74·5	1	76·5	1	75·5	7	78·5	2	78·5	6
72·0	1	74·0	3	76·0	—	74·5	8	77·5	8	77·5	7
71·5	2	73·5	—	75·5	4	73·5	9	76·5	5	76·5	10
71·0	2	73·0	5	75·0	1	72·5	4	75·5	4	75·5	4
70·5	5	72·5	1	74·5	—	71·5	7	74·5	8	74·5	7
70·0	4	72·0	3	74·0	1	70·5	6	73·5	5	73·5	4
69·5	4	71·5	4	73·5	3	69·5	2	72·5	6	72·5	7
69·0	2	71·0	6	73·0	5	68·5	2	71·5	5	71·5	3
68·5	2	70·5	2	72·5	6	66·5	1	70·5	3	70·5	—
68·0	4	70·0	3	72·0	4					69·5	—
67·5	—	69·5	2	71·5	3			66·5	1	68·5	—
67·0	5	69·0	4	71·0	1					67·5	1
66·5	6	68·5	2	70·5	4						
66·0	2	68·0	3	70·0	2						
65·5	2	67·5	2	69·5	5						
65·0	1	67·0	4	69·0	7						
64·5	—	66·5	2	68·5	2						
64·0	—	66·0	1	68·0	2						
63·5	—	65·5	—	67·5	1						
63·0	1	65·0	—	67·0	1						
62·5	—	64·5	—	66·5	—						
62·0	—	64·0	—	66·0	—						
61·5	—	63·5	—	65·5	—						
61·0	1	63·0	1	65·0	—						
				64·5	—						
				64·0	1						

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
86·5	1	88·5	1	94·0	1	100	2	105	1
86·0	—	87·5	4	93·5	1	99	1	104	2
85·5	1	86·5	—	93·0	1	98	—	103	—
85·0	—	85·5	3	92·5	—	97	2	102	1
84·5	1	84·5	6	92·0	—	96	1	101	4
84·0	—	83·5	6	91·5	2	95	2	100	1
83·5	—	82·5	5	91·0	1	94	5	99	2
83·0	3	81·5	7	90·5	—	93	3	98	3
82·5	2	80·5	4	90·0	1	92	9	97	4
82·0	—	79·5	3	89·5	3	91	2	96	5
81·5	3	78·5	5	89·0	1	90	8	95	6
81·0	5	77·5	2	88·5	7	89	3	94	10
80·5	3	76·5	—	88·0	2	88	3	93	2
80·0	3	75·5	—	87·5	1	87	1	92	2
79·5	2	74·5	1	87·0	1	86	1	91	2
79·0	3	73·5	1	86·5	3	85	1	90	1
78·5	3			86·0	5	84	1	89	2
78·0	3			85·5	—	83	—	88	—
77·5	4			85·0	3	82	—	87	—
77·0	2			84·5	3	81	—	86	—
76·5	3			84·0	1	80	1	85	1
76·0	1			83·5	2				
75·5	3			83·0	1				
75·0	3			82·5	1				
74·5	1			82·0	1				
74·0	1			81·5	—				
				81·0	1				
69·5	1								

TABLE XXIII—(*continued*). (*g*) *Weight Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*
11 12	1	14 8	1	16 8	1	18 8	1	20 8	1	21 8	2
11 8	1	14 0	1	16 0	—	18 0	—	20 0	—	21 0	—
11 4	—	13 8	1	15 8	2	17 8	—	19 8	1	20 8	—
11 0	3	13 0	5	15 0	2	17 0	1	19 0	1	20 0	1
10 12	1	12 8	1	14 8	3	16 8	3	18 8	2	19 8	3
10 8	2	12 0	7	14 0	6	16 0	4	18 0	3	19 0	3
10 4	3	11 8	10	13 8	8	15 8	7	17 8	4	18 8	6
10 0	3	11 0	7	13 0	4	15 0	6	17 0	5	18 0	7
9 12	5	10 8	7	12 8	6	14 8	6	16 8	7	17 8	5
9 8	5	10 0	10	12 0	9	14 0	6	16 0	4	17 0	9
9 4	—	9 8	2	11 8	7	13 8	11	15 8	6	16 8	2
9 0	5	9 0	6	11 0	4	13 0	2	15 0	7	16 0	6
8 12	6	8 8	2	10 8	4	12 8	6	14 8	5	15 8	3
8 8	4	8 0	1	10 0	2	12 0	2	14 0	1	15 0	1
8 4	2			9 8	1	11 8	1	13 8	2	14 8	3
8 0	4			9 0	1	11 0	2	13 0	4	14 0	3
7 12	1					10 8	1	12 8	—	13 8	1
7 8	1							12 0	2	13 0	1
7 4	3							11 8	—		
7 0	1							11 0	1		
6 12	1										
6 8	1										
6 4	—										
6 0	1										

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*
23 0	1	26 0	1	24 8	2	26 0	2	28 0	1	30 0	1
22 8	—	25 8	—	24 0	2	25 0	2	27 0	1	29 0	1
22 0	—	25 0	—	23 8	1	24 0	5	26 0	2	28 0	2
21 8	2	24 8	—	23 0	2	23 0	7	25 0	5	27 0	5
21 0	2	24 0	1	22 8	3	22 0	12	24 0	5	26 0	5
20 8	3	23 8	1	22 0	5	21 0	6	23 0	9	25 0	7
20 0	3	23 0	—	21 8	4	20 0	5	22 0	7	24 0	7
19 8	4	22 8	—	21 0	6	19 0	9	21 0	6	23 0	10
19 0	6	22 0	4	20 8	3	18 0	1	20 0	6	22 0	4
18 8	7	21 8	2	20 0	5	17 0	2	19 0	6	21 0	7
18 0	4	21 0	2	19 8	1	16 0	1	18 0	—	20 0	2
17 8	2	20 8	7	19 0	6	15 0	1	17 0	1	19 0	2
17 0	4	20 0	6	18 8	2			16 0	—	18 0	—
16 8	2	19 8	4	18 0	3			15 0	1	17 0	1
16 0	5	19 0	5	17 8	3						
15 8	2	18 8	2	17 0	4						
15 0	1	18 0	2	16 8	—						
14 8	—	17 8	7	16 0	1						
14 0	1	17 0	2	15 8	1						
13 8	1	16 8	3	15 0	—						
		16 0	2	14 8	1						
		15 8	—								
		15 0	1								
		14 8	—								
		14 0	1								

TABLE XXIII—(*continued*).

Month 15		Month 18		Month 24		Month 30		Month 36	
Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*
31 8	1	32 9	2	37 0	1	41 0	1	44 0	1
31 0	—	31 9	2	36 8	—	40 0	—	43 0	—
30 8	1	30 9	3	36 0	—	39 0	—	42 0	—
30 0	—	29 9	2	35 8	—	38 0	1	41 0	2
29 8	—	28 9	2	35 0	—	37 0	2	40 0	—
29 0	2	27 9	3	34 8	3	36 0	2	39 0	2
28 8	1	26 9	10	34 0	—	35 0	3	38 0	1
28 0	—	25 9	8	33 8	—	34 0	1	37 0	3
27 8	5	24 9	2	33 0	2	33 0	2	36 0	1
27 0	3	23 9	6	32 8	1	32 0	6	35 0	4
26 8	2	22 9	6	32 0	3	31 0	7	34 0	7
26 0	1	21 9	1	31 8	—	30 0	6	33 0	6
25 8	3	20 9	—	31 0	1	29 0	2	32 0	4
25 0	4	19 9	—	30 8	2	28 0	6	31 0	5
24 8	4	18 9	1	30 0	—	27 0	4	30 0	5
24 0	6			29 8	2	26 0	1	29 0	5
23 8	1			29 0	4	25 0	—	28 0	1
23 0	3			28 8	5	24 0	1	27 0	—
22 8	2			28 0	3	23 0	—	26 0	1
22 0	2			27 8	2	22 0	1	25 0	—
21 8	5			27 0	1			24 0	1
21 0	2			26 8	3				
20 8	1			26 0	3				
20 0	1			25 8	2				
19 8	1			25 0	2				
19 0	—			24 8	1				
18 8	—			24 0	—				
18 0	—			23 8	1				
17 8	—			23 0	—				
17 0	—			22 8	1				
16 8	1								

TABLE XXIII—*(continued).* *(h) Stem Length Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
39·0	1	40·5	1	42·3	1	44·1	1	46·5	—	47·5	1
38·7	—	40·2	1	42·0	2	43·8	1	46·0	2	47·0	3
38·4	—	39·9	6	41·7	2	43·5	3	45·5	3	46·5	2
38·1	1	39·6	1	41·4	5	43·2	3	45·0	3	46·0	2
37·8	2	39·3	1	41·1	3	42·9	1	44·5	2	45·5	4
37·5	—	39·0	2	40·8	5	42·6	3	44·0	5	45·0	3
37·2	3	38·7	3	40·5	2	42·3	6	43·5	6	44·5	6
36·9	1	38·4	3	40·2	5	42·0	4	43·0	3	44·0	5
36·6	2	38·1	3	39·9	5	41·7	3	42·5	9	43·5	6
36·3	3	37·8	4	39·6	3	41·4	6	42·0	2	43·0	8
36·0	6	37·5	5	39·3	6	41·1	4	41·5	8	42·5	8
35·7	2	37·2	4	39·0	2	40·8	5	41·0	6	42·0	3
35·4	5	36·9	2	38·7	6	40·5	1	40·5	1	41·5	2
35·1	1	36·6	3	38·4	3	40·2	1	40·0	3	41·0	1
34·8	9	36·3	3	38·1	1	39·9	6	39·5	1	40·5	1
34·5	3	36·0	1	37·8	3	39·6	2	39·0	1	40·0	1
34·2	2	35·7	3	37·5	—	39·3	—	38·5	—	39·5	1
33·9	4	35·4	3	37·2	2	39·0	—	38·0	1	39·0	—
33·6	1	35·1	2	36·9	2	38·7	3	37·5	—		
33·3	2	34·8	—	36·6	—	38·4	—	37·0	1		
33·0	1	34·5	1	36·3	1	38·1	—				
32·7	—	34·2	—	36·0	1	37·8	1				
32·4	1	33·9	—	35·7	—	37·5	—				
32·1	—	33·6	—	35·4	1	37·2	1				
31·8	1	33·3	—								
31·5	—	33·0	1								
31·2	—										
30·9	1										
30·6	—										
30·3	—										
30·0	1										

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
48·0	3	49·5	—	50·0	1	51·5	1	52·0	1	52·5	2
47·5	4	49·0	1	49·5	4	51·0	—	51·5	1	52·0	—
47·0	2	48·5	4	49·0	3	50·5	—	51·0	—	51·5	1
46·5	1	48·0	3	48·5	1	50·0	2	50·5	2	51·0	2
46·0	1	47·5	3	48·0	4	49·5	4	50·0	2	50·5	2
45·5	5	47·0	5	47·5	6	49·0	4	49·5	3	50·0	4
45·0	3	46·5	4	47·0	3	48·5	1	49·0	6	49·5	3
44·5	8	46·0	7	46·5	6	48·0	10	48·5	5	49·0	8
44·0	6	45·5	2	46·0	6	47·5	2	48·0	4	48·5	3
43·5	5	45·0	5	45·5	3	47·0	3	47·5	4	48·0	4
43·0	5	44·5	3	45·0	5	46·5	7	47·0	5	47·5	6
42·5	2	44·0	6	44·5	5	46·0	4	46·5	2	47·0	6
42·0	1	43·5	2	44·0	3	45·5	6	46·0	4	46·5	4
41·5	1	43·0	3	43·5	1	45·0	4	45·5	4	46·0	3
41·0	—	42·5	—	43·0	—	44·5	2	45·0	3	45·5	2
40·5	—	42·0	—	42·5	—	44·0	2	44·5	—	45·0	—
40·0	—	41·5	1	42·0	1	43·5	—	44·0	—	44·5	—
39·5	—	41·0	—	41·5	—	43·0	—	43·5	—	44·0	—
39·0	—	40·5	—	41·0	—	42·5	—	43·0	—	43·5	—
38·5	—	40·0	—	40·5	—	42·0	—	42·5	—	43·0	—
38·0	1	39·5	1	40·0	1	41·5	—	42·0	—	42·5	—
						41·0	—	41·5	—	42·0	—
						40·5	1	41·0	1	41·5	—
						40·0	—			41·0	1

TABLE XXIII—(*continued*).

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
55·0	1	54·5	2	57·0	1	58	1	60	1
54·5	—	54·0	2	56·5	—	57	—	59	1
54·0	—	53·5	1	56·0	1	56	2	58	2
53·5	1	53·0	5	55·5	3	55	6	57	5
53·0	2	52·5	2	55·0	—	54	8	56	2
52·5	3	52·0	2	54·5	1	53	10	55	8
52·0	7	51·5	5	54·0	1	52	3	54	12
51·5	2	51·0	2	53·5	5	51	3	53	7
51·0	3	50·5	4	53·0	4	50	5	52	5
50·5	4	50·0	8	52·5	3	49	1	51	5
50·0	2	49·5	3	52·0	5	48	—	50	—
49·5	9	49·0	5	51·5	—	47	—	49	—
49·0	4	48·5	—	51·0	5			48	1
48·5	3	48·0	1	50·5	5				
48·0	7	47·5	—	50·0	2				
47·5	—	47·0	2	49·5	1				
47·0	1	46·5	—	49·0	1				
46·5	1	46·0	—	48·5	—				
46·0	—	45·5	1	48·0	1				
45·5	—								
45·0	—								
44·5	—								
44·0	1								

(*i*) *Shoulder Width Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
16·5	1	17·0	3	17·7	4	19·2	2	21·4	1	21·4	1
16·0	—	16·5	4	17·4	2	18·9	1	21·1	1	21·1	—
15·5	2	16·0	12	17·1	6	18·6	3	20·8	—	20·8	1
15·0	10	15·5	7	16·8	3	18·3	2	20·5	1	20·5	2
14·5	10	15·0	17	16·5	10	18·0	3	20·2	—	20·2	2
14·0	12	14·5	5	16·2	15	17·7	4	19·9	1	19·9	2
13·5	9	14·0	6	15·9	5	17·4	11	19·6	1	19·6	4
13·0	2	13·5	1	15·6	4	17·1	12	19·3	3	19·3	5
12·5	1	13·0	—	15·3	2	16·8	7	19·0	4	19·0	6
12·0	—	12·5	—	15·0	3	16·5	5	18·7	1	18·7	7
11·5	—	12·0	1	14·7	1	16·2	5	18·4	4	18·4	7
11·0	—			14·4	1	15·9	3	18·1	7	18·1	5
10·5	1					15·6	—	17·8	5	17·8	4
						15·3	—	17·5	9	17·5	4
						15·0	—	17·2	6	17·2	1
						14·7	—	16·9	8	16·9	3
						14·4	1	16·6	2	16·6	2
								16·3	2		
								16·0	—		
								15·7	—		
								15·4	1		

TABLE XXIII—(*continued*).

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
22·0	1	22·3	1	22·3	1	23·5	1	22·3	3	22·9	1
21·5	—	22·0	—	22·0	2	23·0	—	22·0	2	22·6	1
21·0	2	21·7	1	21·7	2	22·5	1	21·7	4	22·3	2
20·5	2	21·4	3	21·4	2	22·0	2	21·4	4	22·0	4
20·0	6	21·1	3	21·1	6	21·5	7	21·1	1	21·7	3
19·5	10	20·8	3	20·8	—	21·0	3	20·8	3	21·4	1
19·0	9	20·5	2	20·5	5	20·5	7	20·5	1	21·1	2
18·5	9	20·2	4	20·2	2	20·0	8	20·2	5	20·8	3
18·0	8	19·9	3	19·9	4	19·5	8	19·9	3	20·5	3
17·5	2	19·6	2	19·6	3	19·0	3	19·6	2	20·2	2
17·0	1	19·3	8	19·3	7	18·5	4	19·3	2	19·9	3
16·5	1	19·0	4	19·0	2	18·0	2	19·0	3	19·6	2
		18·7	4	18·7	5	17·5	1	18·7	—	19·3	6
		18·4	3	18·4	5			18·4	—	19·0	1
		18·1	2	18·1	—			18·1	1	18·7	1
		17·8	4	17·8	—			17·8	—		
		17·5	—	17·5	—			17·5	—		
				17·2	—			17·2	—		
				16·9	1			16·9	1		

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
23·5	1	25·9	—	24·5	1	25·5	1	25·4	1
23·2	1	25·6	—	24·0	4	25·0	—	25·1	—
22·9	1	25·3	—	23·5	1	24·5	2	24·8	2
22·6	—	25·0	—	23·0	4	24·0	1	24·5	6
22·3	—	24·7	—	22·5	10	23·5	3	24·2	3
22·0	1	24·4	—	22·0	7	23·0	4	23·9	3
21·7	2	24·1	—	21·5	2	22·5	8	23·6	4
21·4	6	23·8	1	21·0	5	22·0	8	23·3	3
22·1	6	23·5	—	20·5	2	21·5	6	23·0	9
20·8	3	23·2	—	20·0	1	21·0	1	22·7	5
20·5	4	22·9	—	19·5	—	20·5	1	22·4	5
20·2	—	22·6	3			20·0	1	22·1	3
19·9	3	22·3	9			19·5	—	21·8	3
19·6	4	22·0	2			19·0	—	21·5	2
19·3	2	21·7	3			18·5	—		
19·0	2	21·4	4			18·0	—		
		21·1	8			17·5	1		
		20·8	1						
		20·5	2						
		20·2	2						
		19·9	1						

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ENDOCRANIAL DIAMETERS AND INDICES. A NEW INSTRUMENT FOR MEASURING INTERNAL DIAMETERS OF THE SKULL.

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I. Introduction.

The close correlation between form of skull and of brain is so conspicuous a phenomenon that on superficial consideration it might seem unnecessary to make it the subject of a statistical study. When we bear in mind, however, the large bony protuberances that are characteristic for most of the prehistoric skulls, the correlation at once becomes less obvious. And everybody who has tried to make comparisons as to the shape of the brain-case between prehistoric and more recent races of man, or even between recent but remotely related races, will undoubtedly have felt the need for a statistical study of the matter. Still greater will be the need therefore in an examination of the human race's phylogenetic development, where the shape and size of the brain will always be the central point in the work of research and where also the anthropoids must be taken into consideration. The crania of these latter are, as we know, provided with large muscle ridges. In many works on the craniology of the anthropoids it is just this considerable incongruency between the shape of the brain and the outer form of the cranium that has been the technical stumbling-block. In this connection it is sufficient to mention the well-known work of Stephanie Oppenheim: *Zur Typologie des Primatencraniums* (20). It is sought to avoid the difficulties by selecting such points of measurement as will eliminate as far as possible the influence of the thickness of the cranial wall, but it cannot be said that in this way it has been found possible to secure a technique which is quite comparable with that employed in human craniology.

Schwalbe in his famous work (23) on the calotte of *Pithecanthropus erectus Dubois* mentions that the breadth-length index of this fragment cannot be directly compared with the breadth-length index for recent skulls—likewise on account of the cranium's "Aussenwerke" (outworks), as Schwalbe (after Virchow) aptly designates the Torus supraorbitalis and similar formations. Thus already in this work the main stress is laid upon the measurements and indices found on casts of the cranial cavity, partly on casts of the Pithecanthropus fragment, partly on casts of anthropoid crania examined for comparison*. In the following paper we shall deal with other more special problems connected with the greatly varying thickness of the cranial wall,

* *Note:* Later on, in 1902, Schwalbe took up the same question for renewed treatment in his "Beziehungen zwischen Innenform und Aussenform des Schädels," but unfortunately it is mainly the cranio-cerebral topography that is dealt with, i.e. a field that has only indirect interest for anthropology.

Although the anthropology of the brain has in later years taken a prominent place in the work of research—cf. Ariëns Kappers' book on the development of the nervous system, in which the chapter on "Data for an Anthropology of the Brain" occupies one-third of the number of pages (12)—only few

but what has already been said is sufficient to show the need for an instrument that will render possible a direct determination of the three chief diameters of the cranial cavity, without sawing through the skull and without making casts. Over a year ago I therefore addressed myself to T. Gundersen, instrument-maker, Oslo, and submitted the task to him. At that time I was not aware that H. Weinert had in 1925 already described an instrument aiming at the same purpose: "Ein neuer Messzirkel zur Ermittelung von Innenmassen" (30). This instrument, which moreover is also mentioned in the latest edition of Martin's *Lehrbuch der Anthropologie* (16), is later described in connection with a large work on the development of the Sinus frontalis (31). In the same work we are promised a more systematic comparison between the exterior and interior form of the skull—"eine eingehendere Arbeit (liegt) im Manuskript bereits vor" (S. 354), but, so far as I know, this work has not yet been published.

The measurements with the new instrument were already far advanced when I became aware of Weinert's work. Afterwards, in my study of the literature, I met with an article in *Biometrika* dealing in part with the same problems, namely, Hoadley and Pearson's "On Measurement of the Internal Diameters of the Skull in relation I. To the Prediction of its Capacity, II. To the 'Pre-eminence' of the Left Hemisphere" (9). As appears from the title of that paper two main problems are there taken up for discussion. In the first part of the work the correlation of skull capacity with the diameters of the cranial cavity is compared with the correlation of skull capacity and external measurements. In other words, the authors investigate whether it is possible by internal measurements to arrive at a more reliable formula for calculation of capacity. In the second part of the work they take up the old question of the sagittal asymmetry of the cranial cavity and consequently of the brain. The prevailing assumption that the left cerebral hemisphere, both in general and as regards certain functions (*e.g.* the speech centres), occupies a leading position as compared with the right hemisphere is, in Hoadley and Pearson's opinion, not justified by reliable morphologico-biometric observations*. And their investigations

attempts have hitherto been made to deal with the form of the brain in its entirety from an anthropological standpoint. In the literature we can find a number of biometric works on details of the different parts of the brain, but the questions which are of essential importance for anthropology are scarcely even formulated, not to speak of their being answered. And this in spite of the statement made by Klaatsch (14) more than twenty years ago: "Nicht die individuellen Verschiedenheiten der einzelnen Furchen sind es, auf die es ankommt, sondern die Kombinationen von Windungs- und Furchungssystemen, an den Hauptpartien, nicht minder aber auch auf die Gesamtform der grossen Abschnitte und des ganzen Grosshirns." Praiseworthy attempts in this direction have recently been made by the American investigators Weil (29), Pickering (21) and Connolly (7), while Hrdlička's work on the fossae cranii, published as early as 1907 (10), must be mentioned also in this connection.

[* This statement is liable to a misunderstanding on the part of the reader. Pearson and Hoadley found that the length of the right hemisphere in their material was greater than that of the left; an examination of the literature showed that the more reliable investigators found the total right hemisphere to be heavier than the total left. The writers of the paper therefore concluded for the size and weight predominance of the right. But they stated finally that:

We do not consider that this is in any way opposed to the view that—at any rate for some functions—the left hemisphere may predominate, because we do not believe that volitional predominance is of necessity associated with such gross characters as size and weight, *loc cit.*, p. 117. ED.]

of the longitudinal development of the two halves of the cranial cavity point decisively in the opposite direction. In this investigation they deal with a very large and at the same time extremely homogeneous series (739 adult male skulls of the long 26th—30th Dynasties Egyptian Series in the Biometric Laboratory), but, as will be explained more fully later on, my measurements have nevertheless led to the opposite result.

Hoadley and Pearson make use of Weinert's instrument and, in contrast to him, they give a very full description of the technical difficulties, which are by no means inconsiderable and which, moreover, we shall revert to later. The authors, as stated, investigated only male skulls belonging to one race. By extending the investigation to both sexes and to several races it would, of course, be possible to deal with a greater number of problems. Not only should we thereby be enabled for the first time to provide a systematically prepared body of comparative material for other investigators who wish to draw parallels regarding the main form of the brain in anthropoids and in prehistoric or recent races of mankind, but we should also get an opportunity of discussing the question as to whether—and, if so, to what degree—the craniological racial differences which anthropology has demonstrated are accompanied by similar differences in the shape of the brain. Thus it is the *racial* differences that form the chief object of my investigation, and not, as in Hoadley and Pearson's work, the laws applying for the *homo sapiens* as a genus. In consequence of the constantly proceeding interaction between shape of skull and shape of brain during each individual's development it is impossible—at any rate for the present—to establish whether the point of attack for the fundamental factors of heredity is (*a*) on the brain itself, (*b*) on the bones of the skull, or (*c*) on both brain and skull. But it must be possible to ascertain the final result of this mutual interaction, provided we have sufficient material at our disposal. It is clear that the problem can here be dealt with only in broad outline, as we must limit the investigation to certain races. It is furthermore clear that when speaking of brain form and asymmetry of the brain in this paper, I mean so far as it is revealed through a study of the skull cavity. We have to disregard the influence of the meningeal membranes, but this point will certainly be of minor importance. In concluding from the transverse asymmetry of the skull cavity as to the probable transverse asymmetry of the brain, I have made one more assumption which will be mentioned in due place.

It will be understood that my task is closely connected with the racial significance of *the thickness of the cranial wall*, a matter that has received comparatively little attention in craniological literature. And as the variation in wall thickness undoubtedly leads to the *inter-racial* formulae for calculation of capacity giving relatively imperfect results, by extending the investigation to different races we also are led to examine that question.

Among other problems of immediate interest which Hoadley and Pearson do not deal with may be mentioned one connected with the asymmetry of the skull. As is known, Elliot Smith as early as 1907 (25) pointed out that the left upper

occipital fossa is generally somewhat more strongly developed than the right. A difference in this respect was found in 80 % of a rather large series of Egyptian skulls and the author, as is reasonable, considers a greater development of the left posterior lobe of the cerebrum to be the fundamental cause of this. At the same time this phenomenon is deemed to bear relation to the usually occurring greater development of the sinus transversus on the right side; Elliot Smith assumes that there exists here a simple causal relationship: "The fulness of the left occipital pole seems to be the reason for the dextral bending of the superior longitudinal sinus, the flattening of the right pole allowing more room for the bigger sinus on that side" (p. 577). I shall later on take up this question for further discussion. For the present I shall merely mention that M. L. Tildesley (27) from a statistical treatment of a rather large body of material has not been able to confirm Elliot Smith's finding as regards this last-mentioned correlation.

In what has been said above the plan of the present work is made clear. After having first described the instrument and mentioned the technical difficulties—which we hope to find considerably diminished in comparison with Weinert's procedure—we shall discuss the absolute measurements and the three main indices calculated therefrom, with special regard to their anthropological significance. In connection therewith I shall take up the question of the form of the brain in prehistoric man and in the anthropoids, where material for comparison is still unfortunately all too scanty. In Section V I shall deal with the asymmetry conditions of the cranial cavity, my task being here somewhat enlarged as compared with Hoadley and Pearson's, and I shall then finally enter upon the question of capacity calculation, where the chief interest turns upon the possibility of establishing an interracial formula which in accuracy can compare with the best intraracial formulae.

I am fully aware that the inclusion of so many problems renders it difficult to deal thoroughly with each separate question and that the non-homogeneous nature of the material constitutes in certain respects a considerable weakness. Where the results must be regarded as only provisional, that fact will appear from the text, and by constantly taking into account the error of differences (statistically determined), I hope to avoid the dangers involved in the *relative* scantiness of the material.

II. Instrument, Technique, Material, Error of Measurement.

The instrument. For the purpose of comparison it is necessary first to give a very brief description of Weinert's instrument. As shown in Fig. 57 in Weinert's work, this consists of two slightly curved steel rods, the angle between which can be regulated by means of a hinge. Along these rods run two sliders, with the same curve as the rods themselves. After the instrument has been introduced into the cranial cavity the rods are fixed in suitable positions and the sliders are pushed forward until they touch the inside of the skull. It is noted how far the sliders

have been pushed forward, and when the instrument is then taken out it is brought back to the same position as before. The distance between the tips of the sliders is thereupon read off by aid of a millimetre-rule. As it is always a question of maximum measurements, this procedure must be repeated a few times until it is certain that the highest value has been found.

In the new instrument two flexible bands of steel are fixed to a metal bar about 12 mm. wide, which is graduated in millimetres on each side. By aid of a small slide the steel bands can be moved outwards and the distance between their tips is found as the sum of the figures read off on the scale behind the sliders. The "head" of the instrument is provided with a small roller, which ensures that the steel bands go straight outwards, and care is of course taken to preclude any lateral deviation. The bands are given a breadth of 10 mm., but their forward ends are rounded off so that the surface of contact with the bone will be barely 2 mm. wide. The dimensions of the head are made as small as possible, namely, 12×24 mm., a size which renders possible an investigation of the smallest human crania, nor will the majority of anthropoids be likely to offer any difficulties in this respect. Of great importance is the fact that the instrument is provided with a mirror, so that we can inspect the interior of the skull beforehand and during the measuring can see where the tip of the band touches upon the wall. It is of course necessary to light up the cranial cavity by means of a small electric lamp, which is introduced through the foramen magnum. As the skull during measurement lies with base upwards, this presents no difficulties. The skull must be well and firmly supported on a bag filled with mustard-seed or the like. It is sufficient to provide the instrument with one mirror, as the posterior point of measurement can always be viewed directly. I here leave out of consideration the measurement of internal breadth and height, since as regards these dimensions it is of minor importance to be able to see the points of contact (Fig. 1 (*a*), (*b*), (*c*)).

I shall now make a comparison between Weinert's instrument and the one here described. As I personally have no experience of Weinert's instrument, and as his own particulars about the technique are scanty, I must refer to Hoadley and Pearson's above-mentioned paper in *Biometrika*. Speaking of Weinert's publication in the *Anthrop. Anzeiger* (30), they say on p. 85: "No directions for the use of the instrument are provided with it nor in the above paper, only in the latter we find a diagram illustrating what is apparently the median sagittal plane of the skull with the instrument set for measuring the length. In this diagram the maximum foraminal length is almost as great as the distance from the basion to the tip of the dorsum sellae; with such a magnitude of the foramen it is possible to place the hinge of the measuring circle (Weinert's callipers) entirely within the foramen as in the diagram. With crania with more moderate foramina we have not found this possible, and greater difficulty may then arise in taking the internal measurements; it is less easy to grope for the maximum diameter." From what has been said it will appear that the advantages of the new instrument are the

following: (1) there is never any difficulty in entering through the foramen magnum, (2) we can see the points of measurement*, (3) the measurements can be read off direct from the instrument, and (4) we therefore have less difficulty in

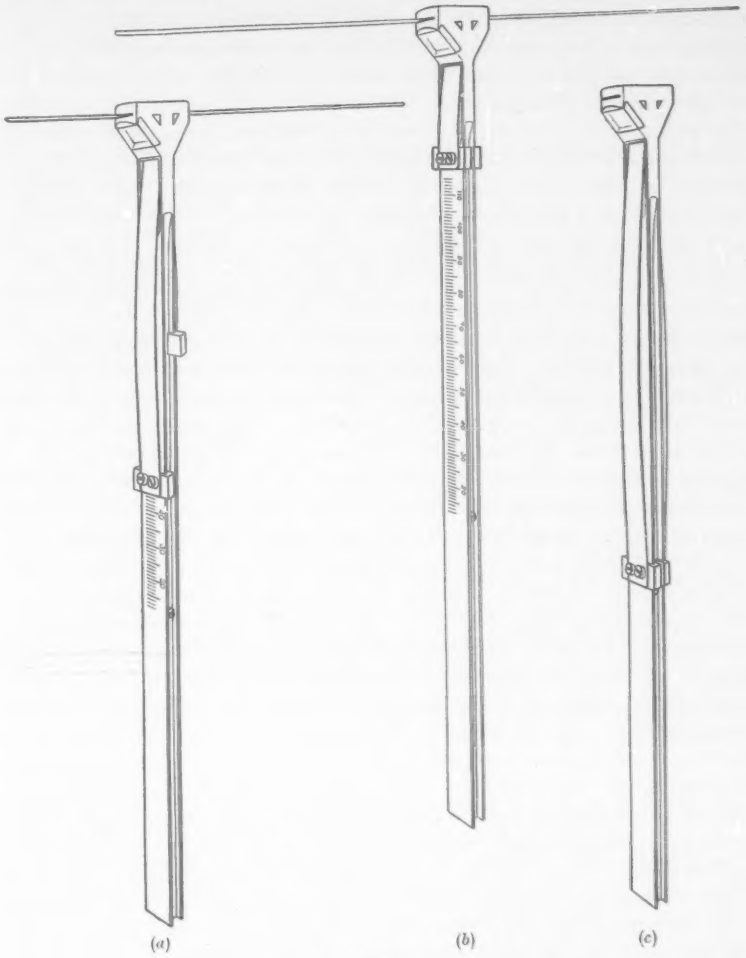

Fig. 1.

adjusting it to obtain the maximum value. On the other hand, it must naturally be admitted that the flexible steel bands have to be handled with a certain degree of caution. Especially, they must never be pushed forward with any great force;

[* Is this wholly in accord with the statement in the last lines of the previous paragraph? ED.]

the ends should just barely touch the walls of the skull. The fear naturally existing, that the bands will bend on meeting a solid wall, proves to be quite unfounded*. Numerous control measurements made with a view to this latter possibility have shown that the instrument's own limit of error is less than $\frac{1}{2}$ mm. in a range of 0 to 200 mm., and as regards our material—and for anthropological material altogether—greater dimensions are hardly likely to come into consideration. It is of course obvious that this error of the instrument itself—which may here be left out of account, since all measurements are read off to the nearest millimetre—must not be confused with the actual error of measurement in determination of the diameters of the cranial cavity, where difficulties occasioned by the peculiar nature of the material also come into consideration. As regards this point, I may refer to the last portion of the present section.

Also as regards the *technique of measurement* in general we may in all essentials refer to Hoadley and Pearson's article. Although this latter was, as stated, unknown to the author when the present material was being investigated, the mode of procedure, so far as it appears from the directions given, has been in principle the same. Of the greatest importance is Hoadley and Pearson's remark on pp. 86—87, that the *maximum longitudinal diameter* of the cavity does *not* lie in the median plane—of which we can easily convince ourselves by examining sectioned skulls—but about 1 cm. to the side thereof, so that the measuring points correspond to the most prominent parts of the frontal lobe or of the posterior lobe. "The reader who will examine a sectioned skull will find that the median sagittal section is ridged to a greater or less extent both anteriorly and posteriorly; we have the crest for the attachment of the falx cerebri, the crista galli of the ethmoid, and the internal occipital protuberance and the sagittal ridge associated with it" (p. 86). For this reason it is practically impossible to take an internal longitudinal measurement in the *median* sagittal plane, and apart therefrom such a measurement would be of no interest, since the purpose of the measurements of the cranial cavity is to eliminate the influence of the bone thickness. But if we are thus obliged to abandon the obtaining of a longitudinal diameter in the median plane, we must on the other hand be careful constantly to have the points of contact for measurement lying strictly sagittally; in other words, the perpendicular distances from the points of contact to the median plane represented by the crista sagittalis must be equal in front and behind, so far as it is possible to secure this by estimation with the eye. To the above directions given by Hoadley and Pearson it should furthermore be added that we must always take care that the posterior point of measurement stands free of the furrow of the sinus sagittalis or the latter's deflection into the sinus transversus. As the sinus transversus is generally more strongly developed on the right side than on the left, it would lead to our

* In order to strengthen the bands the author got Mr Gundersen to make a new instrument where two flexible bands are riveted together on each side of the bar. This new instrument, which has not been used as yet, looks exactly like the former, but the bands are very much strengthened. It might perhaps be possible to combine three bands on each side, thus securing still more strength without reducing their flexibility. [If the "fear" be "quite unfounded" why this need to strengthen the bands? Ed.]

getting too large a mean value for the longitudinal diameter of the right side of the cavity if we were not observant of this source of error. For the sake of completeness it should further be mentioned that—as follows moreover as a corollary from what has been said above—the measuring points for the *maximum breadth* must lie in a transverse plane and, of course, be strictly symmetrical on the two sides. But otherwise we must also here—as in every maximum measurement—take into account the individual peculiarities of the skull. It will very soon be discovered that the maximum breadth of the cavity generally lies in a plane very near to the centre of the occipital condyles, a regularity which is probably not merely fortuitous but due to mechanical factors associated with the upright manner of walking. (Cf. the anthropoids, in which the maximum breadth of the cranial cavity is constantly found considerably *in front of* this plane, provided the skull is orientated in the Frankfurt horizontal. See also Mollison's work, *Statik des menschlichen Schädels* (18). An examination of prehistoric skulls with a view to this question would not be without interest.)

It remains to be mentioned that the two height measurements, *internal basion height* and *internal opisthion height*, have been found by aid of a vertically placed rod, graduated in millimetres, which is shown in Fig. 2 (p. 96). The skull is orientated in the ear-eye plane by means of Mollison's craniophor. We are aware that the furrow of the sinus sagittalis is included in these measurements, but at this place the furrow is so slightly developed and probably varies so little that we may be permitted to disregard its influence. Of the two height measurements most importance is assigned to the first-mentioned, because it comes natural to compare it with the external measurement, basion-bregma height. With a view to the determination of capacity we have also calculated a third height, namely, internal *ear-bregma height*, which has been found indirectly by subtracting from the external measurement the individually determined thickness of the os parietale. This latter value was, it is true, measured at the lower posterior corner of the bone (right side and about 2 cm. above the sutura mastoidea)—through the foramen magnum by aid of ordinary callipers—and not at the bregma, but the difference is probably negligible.

Material. As to the material I need only say that it belongs to the Anatomical Institute in Oslo, the Principal of which, Professor K. E. Schreiner, has not only been so kind as to place the skulls at my disposal, but has even allowed me to make use of material assembled for another purpose. This latter concerns Norwegians and Lapps. A systematic treatment of the whole material of Lapp skulls, far more comprehensive than that dealt with here, now lies completed from Schreiner's hand (22). Although I thus, from considerations of time, have not included all the skulls of Lapps and Norwegians that stood at my disposal, yet I have not made any selection, except in so far as only adult skulls in good preservation have been employed. The two groups consisting of Australians and Maori will form the foundation for another work by myself, already commenced, on the craniology of the Polynesians.

Originally the material consisted of 328 skulls of Norwegians, Lapps, Maori and Australian aborigines. After I had finished the study, I measured 41 Eskimos to check some of the more doubtful results. The means, the σ's and v's, for all

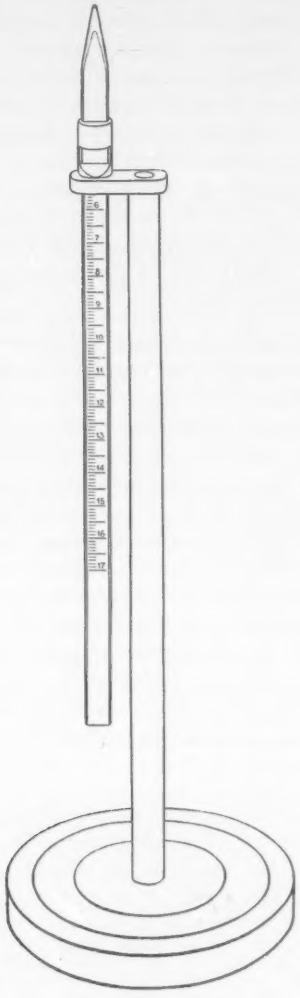

Fig. 2.

the measurements are given in Table XIII. I have also measured 10 anthropoid skulls, likewise belonging to the Anatomical Institute. These few skulls will serve to demonstrate that the same technique may be applied to this material.

It now remains to set forth a few figures illustrating *the accuracy of measurement*. What is of interest here is the error of measurement in determination of the length of the cranial cavity, where the difficulties are greatest. As described in detail by Hoadley and Pearson (p. 111), this error can be determined in the following manner: We take two sets of independent measurements and calculate $\frac{1}{2}(L_L + L_R)$ for both. If the individual observations in the first and second measurements are designated by z_1 and z_2 respectively, then $\sigma_{z_1 - z_2} \times 0{\cdot}67449$ will give "the probable error of a single measurement."

In our own material, this value was found to be 0·7406 mm. ($\sigma = 1{\cdot}098$ mm.) for 324 skulls, since 4 skulls were not available for the second measurement, whereas Hoadley and Pearson find the error to be 1·2043 mm., with use of Weinert's instrument.

We see that the error of measurement is reduced considerably (by as much as 38 %), which is not an improbably great reduction when we consider the advantages of our instrument, especially the fact that the points of measurement can be seen. With greater practice I regard it as possible to reduce the probable error to less than 0·5 mm.

III. The Anthropological Importance of Internal Measurements and Indices. Thickness of the Roof of Skull. Variability and Correlation.

As mentioned in the Introduction, our first object is to investigate to what extent the craniological differences of race expressed by the three main diameters, and the indices computed on the basis thereof, are accompanied by parallel differences in the form and size of the brain. Table I gives a survey of the most important absolute measurements. Here, as in the other tables, m denotes "standard error," i.e. $\frac{\sigma}{\sqrt{n}}$. Where "probable error," i.e. $0{\cdot}67449m$, is used, this fact is specially indicated.

Norwegians and Lapps are relatively well represented, while the values for Maori and Australians can only be taken as provisional, without this fact having necessarily any great influence as to the questions here dealt with. It is seen that the difference in sex comes well into evidence, both in the external and in the internal values. The maximum length of the cranial cavity (length of the cerebrum) is in women from over 3 to about 7·5 mm. less than in men, while the breadth is from about 4·5 to about 6 mm. and the height from about 3 to 6 mm. less in women than in men. These differences are from 2 to 9 times their standard errors, only the difference in height of brain-box in Australians being statistically very doubtful (less than twice the standard error of difference). Further, it is clearly evident from the *internal* measurements that the Australians' brain is not only very narrow but also extremely short, while on the other hand it shows considerable height, not only relatively but also absolutely regarded. Before proceeding to illustrate these points by means of indices we shall see what Table I has to tell about the *bone thickness* in our 4 groups. Subject to the reservation that an

TABLE I.

External measurements	Norwegians $M \pm m$ (standard error)		Lapps $M \pm m$		Maori $M \pm m$		Australians $M \pm m$	
	♂, $n=81$	♀, $n=76$	♂, $n=63$	♀, $n=55$	♂, $n=16$	♀, $n=14$	♂, $n=13$	♀, $n=10$
1. Maximum length	187·70 ± ·65	179·28 ± ·67	179·17 ± ·86	170·96 ± ·71	186·81 ± 1·04	180·00 ± ·95	183·30 ± 1·21	173·40 ± 1·69
2. Maximum breadth	142·34 ± ·56	135·37 ± ·45	146·52 ± ·62	142·05 ± ·59	137·88 ± 1·15	132·64 ± ·97	130·85 ± 1·27	124·60 ± 1·10
3. Basion-bregma height	132·31 ± ·73	125·34 ± ·54	129·05 ± ·57	124·05 ± ·58	136·63 ± 1·15	130·21 ± 1·03	135·92 ± 1·26	130·70 ± ·95
Internal measurements								
4. Maximum length of cavity $\frac{R+L}{2}$	172·28 ± ·69	166·09 ± ·68	168·05 ± ·83	161·57 ± ·63	171·21 ± 1·06	168·04 ± 1·17	163·77 ± 1·07	156·40 ± 1·69
5. Maximum breadth of cavity ...	134·05 ± ·49	128·22 ± ·45	140·05 ± ·61	135·24 ± ·56	131·31 ± 1·12	126·57 ± 1·03	123·38 ± 1·18	117·60 ± ·96
6. Internal basion height... ...	127·07 ± ·72	121·03 ± ·52	124·88 ± ·58	120·62 ± ·55	130·44 ± ·94	125·57 ± 1·05	128·77 ± 1·42	125·90 ± 1·34

indirect determination of bone thickness can yield only approximately correct figures* and that the upper point of measurement for No. 6 does not correspond exactly to bregma (No. 3), ½ (2—5) will give the bone thickness at euryon and 3—6 the approximate thickness at bregma.

The values given below (Table II) are, however, the means of the individually computed differences (not the differences between the means)†, whereby it becomes possible to take the standard errors also into account. For comparison with the computed bregma thickness, the thickness of the os parietale (right side) at the lower posterior corner is also given, measured directly through the foramen magnum.

TABLE II.

	Norwegians		Lapps		Maori		Australians	
	♂ $n=81$	♀ $n=76$	♂ $n=63$	♀ $n=55$	♂ $n=16$	♀ $n=14$	♂ $n=13$	♀ $n=10$
Thickness at euryon	4·15 ±0·09	3·57 ±0·10	3·24 ±0·11	3·41 ±0·12	3·28 ±0·14	3·04 ±0·20	3·73 ±0·30	3·50 ±0·22
Thickness at bregma (computed)	5·25 ±0·20	4·32 ±0·18	4·16 ±0·19	3·44 ±0·23	6·19 ±0·53	4·64 ±0·47	7·15 ±0·64	4·80 ±0·77
Thickness of os parietale (direct)	5·33 ±0·17	4·38 ±0·13	4·30 ±0·15	3·98 ±0·14	6·88 ±0·35	5·79 ±0·32	7·39 ±0·56	5·70 ±0·32

We see that the difference of the directly and the indirectly determined thickness of the os parietale is *in general* small, being in 4 groups less than ½ mm. In Lapp women, Maori men, Maori women and Australian women the difference is, however, not inconsiderable. In the two latter groups the computed bregma thickness is probably about 1 mm. too small, which may easily be explained by some irregular contours, where immediately behind the bregma the cranial roof has risen relatively much.

In the next place it is seen that the skull-cap is, as might be expected, somewhat thicker at the bregma than at the euryon and that the male skulls are in general a little thicker than the female. As regards the racial differences revealed in Table II, it is natural to compare Lapps with Norwegians and Maori with Australians. As might *a priori* be expected, the values for Lapps are lower than the values for Norwegians and those for Maori in general lower than for Australians. The difference in the former case may be regarded as statistically certain. If we,

* Owing to the small dimensions here involved we must, in an examination aiming *directly* at the thickness of the skull roof, demand a greater accuracy of measurement than 1 mm., and measurements preferably taken with an instrument specially constructed for the purpose. Such an instrument was (according to Martin) designed by Péan as early as 1896.

[† If the two are not the same, this must be a fault of arithmetic, not of measurement. ED.]

for example, compare the thickness at the euryon in male Lapps (3·24) and in male Norwegians (4·15), the difference is seen to be 0·91 mm., or 6·5 times the standard error of the difference (0·14). The difference between Maori and Australians, on the other hand, is statistically less well founded, but it can hardly be doubted that in a larger body of material the difference would be well beyond the statistical error. It would be of interest to ascertain whether the Australian skull, the thickness of which at the *bregma* is very considerable, is *not* thicker at the euryon than the skulls in an average European material, as seems to appear from Table II.

Our figures accord well with what others have found: Hoadley and Pearson state the thickness at the euryon to be 3·52 mm. and at the bregma* 5·46 mm. (male Egyptians). Todd (28), as the average of 448 sectioned "white crania" (male), found by direct measurement 3·56 mm. at the euryon and 5·88 mm. at the vertex, figures which, together with ours, confirm on the whole Beddoe's table of the values that must be deducted from the external breadth and external height in order to arrive approximately at the internal dimensions (see Martin, p. 726). Only we must make the reservation that the variation—as Todd points out—is so considerable that the table can be applied only to mean values.

We now come to *the thickness of the frontal bone at the glabella.* In order to find this value we must from the difference between external and internal length $\left[L - \frac{L_R + L_L}{2}\right]$ deduct the thickness of the os occipitale at the posterior point of measurement $\left[\frac{O_R + O_L}{2}\right]$. This latter dimension, which has been found directly, is likewise to be seen from the following table. (We have to disregard the fact that the anterior point of measurement for the internal length does not exactly correspond to the glabella-point from which the external length has been measured, but this circumstance will certainly have no disturbing influence on the results.)

TABLE III.

	Norwegians		Lapps		Maori		Australians	
	♂	♀	♂	♀	♂	♀	♂	♀
Thickness of os occipitale	4·21 ±0·11	3·96 ±0·13†	3·01 ±0·13	3·06 ±0·14	4·89 ±0·24	4·54 ±0·25	5·19 ±0·39	5·42 ±0·31
Thickness of os frontale at glabella	11·22 ±0·33	9·15 ±0·27	8·11 ±0·40	6·33 ±0·34	10·71 ±0·52	7·43 ±0·45	14·35 ±0·83	11·58 ±0·74

[* Hoadley and Pearson measured the bone-thickness vertically above the basion, which is not the bregmatic thickness. Ed.

† This is the value given in Table XIII, but if the entries in that table from which it is deduced be correct, it should be 9·23, which will modify the value for Norwegian – Lapps given in the following paragraph from 2·82 to 2·90. Ed.]

Taking first the occipital bone, we again note that the Lapp skulls are the thinnest and the Australian skulls the thickest, although the racial differences are not particularly pronounced. As before, the difference between Norwegians and Lapps is statistically well established, being for the male skulls 1·20 mm., or 7 times the standard error of difference, while the difference between Maori and Australians is statistically doubtful. In males it is 0·30 mm. ($m_{\text{Dif.}} = 0{\cdot}48$ mm.). Further there is seen to be a strikingly small difference between the two sexes. In Lapps and Australians the difference found actually goes in the opposite direction to what should be expected. Quite different is the situation as regards the glabella-thickness. Here the difference between the sexes goes in the same direction in all 4 groups, and lies between 1·78 and 3·28 mm., $m_{\text{Dif.}}$ lying between 0·43 and 1·11. The sexual difference is from 2 to 5 times the standard error of difference. And if we further compare Norwegians with Lapps and Australians with Maori, we find, to take the male skulls first, a difference between the former two of 3·11 mm., and between the latter two of 3·64 mm., that is to say, respectively, 6 and 4 times $m_{\text{Dif.}}$ (0·52 and 0·98). The female skulls give quite similar figures: Norwegians – Lapps 2·82 mm., or 6 times $m_{\text{Dif.}}$ (0·43); Australians – Maori 4·15 mm., or 5 times $m_{\text{Dif.}}$ (0·87). My figures for the glabella-thickness thus also accord very well with earlier findings: Todd's material (male "white crania") gave 11·26 mm. (cf. our Norwegians) and Hoadley and Pearson's male Egyptian skulls showed a difference between external and internal length of 15·10 mm., whereof the author's estimate $\frac{3}{5}$ to be frontal and $\frac{2}{5}$ occipital, that is to say, a glabella-thickness of about 9 mm. (lying between Lapp and Maori).

The marked sexual and racial difference in the glabellar region which I have here demonstrated is not surprising. In the first place, a well-developed torus supraorbitalis is one of the features to which most importance is attached in determination of sex, and as regards the Australians, Mollison (17) has emphasised the great development of the glabella particularly in contrast to the Maori. The flat glabella in Lapps is likewise a well-known phenomenon, which moreover we have seen is accompanied by generally thin skull-bones. In connection with my investigations of the thickness of the skull-wall at the euryon and bregma we can, however, from the figures here given draw certain conclusions of interest for our main subject: the correlation between form of brain and of skull.

I. Since on going over to internal measurement the length is reduced much more than the breadth and the breadth, again, more than the height, all three internal indices will probably on the average show larger values than the corresponding outer indices. The differences here concerned recur again in all 8 groups and they are so considerable that this rule may probably be deemed to apply, if not to all, at any rate to most races. The increased breadth-length index has repeatedly been observed in casts of the cranial cavity, cf. Schwalbe (23), who as early as 1899 could state, on the basis of examination of individual skulls: "Es zeigt sich, dass der Schädelinnenraum des Erwachsenen, wie bei der Mehrzahl der Affen, einen grösseren Index besitzt, als der aussen gemessene Schädel" (S. 36).

II. Even though the same tendency is present in all groups, it is not likely to make itself equally evident in all of them. In the races which show the greatest development of the glabella region the increase in the breadth-length and height-length indices will be most pronounced—in our material especially in the Australians.

By a more searching examination of my figures it would undoubtedly be possible to draw other and more detailed conclusions as to the probable "brain-indices," but we can attain the same object in a simpler manner by a direct comparison of external and internal indices, as in Table IV.

TABLE IV.

External indices	Norwegians		Lapps		Maori		Australians	
	♂	♀	♂	♀	♂	♀	♂	♀
Breadth-length index	75·88 ±0·33	75·56 ±0·31	81·90 ±0·52	83·17 ±0·49	73·87 ±0·73	74·07 ±0·66	71·38 ±0·51	71·92 ±0·83
Height-length index	70·46 ±0·40	69·96 ±0·34	72·05 ±0·39	72·62 ±0·39	73·15 ±0·46	72·35 ±0·50	74·21 ±0·64	75·40 ±0·60
Height-breadth index	93·08 ±0·58	92·67 ±0·52	88·19 ±0·55	87·39 ±0·47	99·23 ±1·22	98·26 ±1·13	103·99 ±1·22	105·04 ±0·82
Internal indices								
Breadth-length index	77·56 ±0·37	77·24 ±0·29	83·46 ±0·50	83·77 ±0·46	77·12 ±0·71	75·36 ±0·64	75·35 ±0·62	75·27 ±0·90
Height-length index	73·89 ±0·45	73·84 ±0·39	74·45 ±0·45	74·72 ±0·42	75·94 ±0·73	74·74 ±0·52	78·67 ±0·96	80·61 ±1·21
Height-breadth index	94·82 ±0·64	94·48 ±0·54	89·40 ±0·56	89·37 ±0·51	99·28 ±1·22	99·30 ±1·14	104·48 ±1·48	107·05 ±0·71

On passing to internal measurement the breadth-length index is increased by from 0·60 to 3·97 index-units, the height-length index by from 2·10 to 5·21 and the height-breadth index from 0·05 to 2·01 index-units. These figures accord very well with what might be expected from our knowledge as to skull thickness. As regards the last-named index the increase is sometimes insignificant, and in individual cases there will of course frequently be found a deviation in the opposite direction, but for the material as a whole the increase must, in spite of the relatively large mean errors, be regarded as a fact, since the same tendency is met with in all 8 groups. Meanwhile, if we wish to examine whether the anthropological differences in shape are lessened or further increased on passing to internal measurements, it is chiefly the first two indices that will be of importance. The breadth-length index has in the highly dolichocephalic Australians increased

by 3·97 and 3·35 index-units for males and females respectively, in the Maori by respectively 3·25 and 1·29 index-units, in Norwegians (mesocephalic on the border of dolichocephaly) by 1·68 and 1·68 index-units, and in the brachycephalic Lapps by only 1·56 and 0·60 units respectively for males and females. Herein we see a distinct tendency to reduce the racial differences, once again as a result of the formation of the glabella,—the increase being also as a rule greater in males than in females.

As regards the height-length index the figures suggest a somewhat different situation. Named in order of succession from the partly hypsicephalic Australians to Norwegians on the border of chamaecephaly the following values are found for the index increase: Australians respectively 4·46 and 5·21, Maori 2·79 and 2·39, Lapps 2·40 and 2·10, Norwegians 3·43 and 3·88. The racial difference between Norwegians and Lapps is lessened on adopting internal measurement, while as between Australians and Maori it is increased. The strongly developed glabella of the Australians thus partially conceals the considerable relative development in height of brain. Had our material included a chamaecephalic type with flat glabella (if such a type exists), as well as a hypsicephalic group with strongly developed glabella, the increase in racial difference would have become still more clearly manifest.

The results of these investigations of shape of skull as contrasted with shape of brain* can thus be provisionally summed up as follows:

1. All three brain-indices have shown increase as compared with the indices for the skull. The difference is greatest in the height-length index, less in the breadth-length index and least in the height-breadth index.

2. The racial differences are constantly less clearly manifest in the breadth-length index of the brain than in that of the skull. The general validity of this rule is of course contingent on a rather strong negative (spurious) correlation between glabella-thickness and external length-breadth index. In my own material I have found the following figures for the coefficient of correlation (r) between the properties mentioned:

Norwegians, male, $-0{\cdot}2960 \pm 0{\cdot}101$; female, $-0{\cdot}2126 \pm 0{\cdot}110$.

Lapps, male, $-0{\cdot}2856 \pm 0{\cdot}117$; female, $-0{\cdot}2621 \pm 0{\cdot}126$.

Interracially this negative correlation will undoubtedly be still more pronounced.

3. On comparing a hypsicephalic race having a relatively strongly developed glabella with a chamaecephalic race with relatively flat glabella, the height-length index for the brain will show greater racial differences than the skull index.

Rules 1 and 3 need no further substantiation from our knowledge of the thickness of the skull roof at the glabella, euryon and bregma.

If we are to have any hope of penetrating more deeply into our problem of the relation between shape of skull and shape of brain, we must also take into considera-

* Let it be mentioned again—to avoid misunderstandings—that "shape of brain" is here used as synonymous with shape of skull cavity.

tion *the conditions of variation and correlation.* What is the chief interest for us here is to compare the distributions of the external with the corresponding internal dimensions (Table V).

Hoadley and Pearson find that the internal length is "considerably more variable than the external length, while in the case of the breadths and heights the difference in the coefficients is negligible, being well within the probable error." In our material there is found no conformity to rule in this respect; sometimes the internal length, sometimes the internal breadth or height is more variable than the external. The difference as a rule is slight, and for each group taken separately does not offer much to build upon (cf. the "standard error" of standard deviation), but such difference as is found generally tends in the direction that the internal measurement is more variable. If we keep to the variation coefficients and count together the + deviations for the brain, we find that 32 out of 48 point in the direction of the brain being more variable and 15 out of 48 in the opposite direction, while 1 shows no difference. On examining the absolute measurements and the indices separately the following figures are obtained: 15 +, 8 – and one 0 deviation for the absolute measurements, and 17 + and 7 – for the indices, and of these latter 7 – deviations five fall to the internal breadth-length index, while the height-length index of the brain is in all eight groups more variable than that of the skull. In all cases it can be concluded from these figures that at any rate *we must reckon with an (intraracial) variation of shape and size of the brain which is not less than that of the skull.* This result is surprising in so far as we should have expected the considerable variation in thickness of the skull to make itself strongly felt. An investigation of the correlation between internal measurement and bone thickness supplies the explanation; for Hoadley and Pearson have shown that the bone thickness is negatively correlated with the corresponding internal measurement, $r_{L_i/(L-L_i)}$, thus giving – 0·36448. A similar correlation is found in our material.

As appears from Table VI (p. 106) the thickness as represented by the glabella-thickness is in all four groups negatively correlated with the length of the cranial cavity. The accompanying errors are, it is true, so large that we must content ourselves with a summary consideration, but for the material as a whole hardly any doubt can prevail as to the significance of this negative correlation, especially in view of Hoadley and Pearson's investigations. We also find that the euryon-thickness in two groups is negatively correlated with the internal breadth, while in two groups the correlation is without significance.

The most natural explanation of these negative correlations is the following: either (*a*) the roof of the skull exerts an inhibitive influence on the growth of the brain, or (*b*), conversely, a relatively considerable growth of the brain will have a reducing effect on the thickness of the cranial roof, or (*c*) both factors come into play at the same time. It is not possible on the basis of the available data to penetrate more deeply into the causative relations. And neither is an investigation of the correlation between bone thickness and the three main indices of the brain particularly illuminating in this respect. As is seen, all these correlations (bone

TABLE V. *Variation of External and Internal Measurements.*

External measurements	Norwegians				Lapps				Maori				Australians			
	♂		♀		♂		♀		♂		♀		♂		♀	
	$\sigma \pm$	v	$\sigma \pm$	v	$\sigma \pm$	v	$\sigma \pm$	v	$\sigma \pm$	v	$\sigma \pm$	v	$\sigma \pm$	v	$\sigma \pm$	v
Maximum length	5·86 ±·460	3·12	5·81 ±·471	3·24	6·79 ±·679	3·79	5·23 ±·499	3·06	4·14 ±·732	2·22	3·55 ±·671	1·97	4·36 ±·855	2·38	5·33 ±1·192	3·07
Maximum breadth	5·03 ±·390	3·53	3·94 ±·320	2·91	5·00 ±·445	3·41	4·39 ±·419	3·09	4·58 ±·810	3·32	3·64 ±·688	2·74	4·57 ±·896	3·49	3·47 ±·775	2·78
Basion-bregma height	6·55 ±·515	4·95	4·75 ±·385	3·79	4·52 ±·403	3·50	4·29 ±·409	3·46	4·61 ±·814	3·37	3·84 ±·726	2·95	4·55 ±·892	3·35	2·99 ±·669	2·29
Breadth-length index	2·95 ±·232	3·89	2·69 ±·218	3·56	4·16 ±·371	5·08	3·65 ±·348	4·39	2·90 ±·513	3·93	2·48 ±·469	3·35	1·85 ±·363	2·59	2·61 ±·584	3·63
Height-length index	3·64 ±·286	5·17	2·95 ±·239	4·22	3·09 ±·275	4·29	2·89 ±·276	3·99	1·85 ±·327	2·53	1·86 ±·351	2·57	2·31 ±·453	3·11	1·91 ±·427	2·53
Height-breadth index	5·26 ±·413	5·56	4·53 ±·367	4·89	4·34 ±·387	4·92	3·49 ±·333	3·99	4·86 ±·859	4·90	4·23 ±·799	4·30	4·39 ±·861	4·22	2·59 ±·579	2·47
Internal measurements																
Maximum length $\frac{R+L}{2}$	6·24 ±·490	3·62	5·90 ±·479	3·55	6·56 ±·584	3·90	4·65 ±·443	2·88	4·25 ±·751	2·48	4·37 ±·826	2·60	3·85 ±·755	2·35	5·35 ±1·196	3·42
Maximum breadth	4·44 ±·349	3·31	3·90 ±·316	3·04	4·88 ±·435	3·48	4·18 ±·398	3·09	4·46 ±·788	3·40	3·87 ±·731	3·06	4·25 ±·833	3·44	3·04 ±·680	2·59
Internal basion height	6·48 ±·509	5·10	4·49 ±·364	3·71	4·64 ±·413	3·72	4·10 ±·391	3·39	3·74 ±·661	2·87	3·93 ±·743	3·13	5·12 ±1·004	3·98	4·25 ±·950	3·38
Internal breadth-length index	3·29 ±·258	4·24	2·57 ±·208	3·33	3·96 ±·353	4·74	3·43 ±·327	4·09	2·85 ±·504	3·70	2·39 ±·452	3·17	2·24 ±·439	2·97	2·85 ±·637	3·79
Internal height-length index	4·02 ±·316	5·44	3·43 ±·278	4·65	3·57 ±·318	4·80	3·14 ±·299	4·20	2·90 ±·513	3·82	1·96 ±·370	2·62	3·45 ±·677	4·39	3·83 ±·856	4·75
Internal height-breadth index	5·79 ±·455	6·11	4·69 ±·380	4·96	4·46 ±·397	4·99	3·75 ±·358	4·20	4·87 ±·861	4·91	4·26 ±·805	4·29	5·33 ±1·045	5·10	2·24 ±·501	2·09

TABLE VI. *Correlations.*

Correlated Pair	Norwegians		Lapps	
	♂ $n=81$	♀ $n=76$	♂ $n=63$	♀ $n=55$
Glabella-thickness & internal length	−·3444±·098	−·2961±·105	−·2354±·119	−·1467±·132
Glabella-thickness & internal breadth	−·2442±·104	−·2407±·108	−·2283±·119	−·0751±·134
Glabella-thickness & internal basion-height ...	−·0431±·111	−·1665±·112	+·0168±·126	−·0858±·134
Euryon-thickness & internal breadth	+·0079±·111	−·2145±·110	−·1617±·123	−·0762±·134
Glabella-thickness & internal breadth-length index	+·0870±·102	+·1027±·113	+·0435±·126	+·0743±·134
Glabella-thickness & internal height-length index	+·1791±·108	+·0421±·115	+·2198±·120	+·0396±·135
Glabella-thickness & internal height-breadth index	+·0942±·110	+·0236±·115	+·1521±·123	−·0304±·135
Euryon-thickness & internal breadth-length index	−·0486±·111	−·0284±·115	+·0694±·125	−·0097±·135
Internal breadth-length index & capacity ...	−·0357±·111	−·2879±·105	−·1319±·124	−·0068±·135
Internal height-length index & capacity	+·0216±·111	−·2227±·109	−·2238±·120	−·0410±·135
Internal height-breadth index & capacity ...	+·1393±·109	−·0434±·115	−·0216±·126	−·0751±·134
Internal length & internal breadth	+·4118±·092	+·4886±·087	+·1318±·124	+·0540±·134
Internal length & internal height	+·2303±·105	+·3563±·100	+·2135±·120	+·1269±·133
Internal breadth & internal height	+·1083±·110	+·0597±·114	+·0276±·126	+·0724±·134

thickness being represented at glabella and euryon) are very slight, only three of them exceeding $2m_r$. Neither has an examination of the correlation between brain-shape and *capacity* led to positive results. It is true that the correlation between capacity and breadth-length index is negative in all four groups, but only in one case, Norwegian females, is it of statistical value in relation to m_r. We must content ourselves with the negative conclusion that, *judging from the material here dealt with*, neither the cranial capacity nor the thickness of the roof of the skull has any essential predetermining influence on the form of the brain. Meanwhile we must bear in mind that an investigation of intra-racial correlations in a comparatively small body of material, such as mine, cannot be decisive as regards this question. I may mention in this connection that my Australian skulls were strikingly thin at the euryon, and the same was the case with the 41 Eskimo skulls (18 male, 23 female) which were examined after the main material had been dealt with (see Table XIII, no. 6 (*c*)). As we know, the Eskimo skulls are also excessively dolichocephalic. If it should prove to be the case that we are here confronted by a general law, namely, that the skulls of the dolichocephalic races are relatively thin at the euryon, then it must be permitted to draw the conclusion that a pressure from

the side wall of the skull is an essential factor in the shaping of the dolichocephalic brain. This conclusion is of all the more interest because it diverges greatly from the generally adopted answer to a much disputed question.

The last three series of coefficients for the relation between the three main diameters of the cranial cavity likewise confirm very well Hoadley and Pearson's results. There is a very strong positive correlation between maximum length and maximum breadth and likewise between maximum length and internal basion-height, but between breadth and height the correlation is practically zero. The positive correlation between the two first-named values suggests the presence of general factors of size (genetic or epistatic) affecting both diameters. From the fact that the correlation is by no means absolute we may conclude that there exist at the same time special shape-determining factors which act more strongly on one diameter than on the other, and such shape-determining factors create an antithesis in the relation between height and breadth of brain which is here sufficiently strong to mask the effect of the general size-factors.

IV. Shape of the Brain in Prehistoric Man and in the Anthropoids.

It is natural to compare the above figures for the three main indices of the brain with the values previously found from casts of the cranial cavity in prehistoric specimens. As early as 1899 Schwalbe (23) compared the internal breadth-length index in the Pithecanthropus, Neanderthal-Kalotte, Spy I and Spy II with that of some isolated recent skulls. None of these prehistoric specimens showed exceptional values: Pithecanthropus 80·0 (in measurements by Dubois and Weinert (31)), Neanderthal-Kalotte 79·1 (after other measurements 78·2 – 80·0), Spy I 75·9, Spy II 81·3*. Further can be mentioned: La Chapelle-aux-Saints 78·3 (Boule and Anthony (4), p. 134), Rhodesia 79·4 (Hrdlička (11), p. 121), Gibraltar 79·0 (Hrdlička, p. 169, "close to"), Ehringsdorf 79·0 (Hrdlička, p. 239, after Weidenreich's "reconstructed 1925 skull"), La Quina 73·8 (Hrdlička, p. 291), Le Moustier 76·7 (Hrdlička, p. 301, after Weinert's reconstruction "close to").

Of the prehistoric finds here mentioned *only the La Quina has had a relatively narrow brain as compared with our recent material.* All the other figures lie above the average for our Australians and not far from the mean values for recent mesocephalic Europeans.

Schwalbe also established the fact that the brain of *the anthropoids* was relatively broad, and the figures given by him for breadth-length index have on the whole been confirmed by later investigators: Chimpanzee 85·5, Gorilla ♀ 84·5, Orang ♀ 96·0. Unfortunately Weil's (29) technique is so different from Schwalbe's and ours that we cannot make use of his measurement of "endocranial casts," and Connolly's (7) excellent paper deals with fixed brains, but it can nevertheless be regarded as established that the orang-outang is farthest removed from man as regards this index, next comes the chimpanzee, while in the gorilla the index partly coincides

* It is not quite clear from Schwalbe's work whether the two Spy figures have been based on casts of the cranial cavity. At any rate, according to Schwalbe they give "die Innenform des Schädels."

with the mean figures for recent and prehistoric man. The considerable variation found especially in the gorilla (different species?) has been pointed out by several investigators, Keith (13), Bolk (2), Harris (8). Keith (p. 78) gives the following figures for endocranial breadth-length index:

Chimpanzee ♂ 84·5, ♀ 84·1 (var. 79 – 92, $n = 19$).
Gorilla ♂ 78·8, ♀ 81·3 (var. 72 – 88, $n = 47$).

In our own material we have found the following values:

Chimpanzee: 81·9.
Gorilla: 79·5 – 82·8 – 84·2 – 82·4 – 82·2.
Orang: 85·7 – 90·2 – 87·8 – 84·1.

None of the three great anthropoids lies entirely outside of the range of variation for man!

With respect to *relative height*, it is a well-known fact that most prehistoric skulls have a height-length index which is far below that of recent human skulls and in a manner forms a bridge between homo sapiens and the great anthropoids. Here, however, we still lack reliable figures which could give us an idea of the situation as regards the brain.

For the *La Chapelle-aux-Saints* skull Boule and Anthony give an internal height-length index of

$$\left(\frac{\text{hauteur basilo-bregmatique} \times 100}{\text{longueur max.}}\right) = 68{\cdot}1.$$

Rhodesia. 70·6 (computed from Hrdlička's figures: endocranial basion-bregma-height 120, maximum length of cerebral hemisphere 170).

Gibraltar. 70·6 ("close to," computed from Hrdlička's figures for greatest length of brain 163 and basion-bregma height less thickness of frontal bone near bregma 115).

Le Moustier. 75·0 (after measurements of Weinert's reconstructed model skull, internal length 176, internal basion-height 132).

It must be deemed surprising that these values also lie within the range of variation for our recent material. As regards the Chapelle-aux-Saints skull the index is, no doubt, very low, but no fewer than 6 of our 81 male Norwegian skulls show still lower values. Neither can the internal height-breadth index be said to be exceptional in the above-mentioned skulls: La Chapelle-aux-Saints 86·9, Rhodesia 88·9, Gibraltar 89·8, Le Moustier 97·8. In this respect we find that 4 of the aforesaid 81 recent crania lie below the lowest figure, that for the La Chapelle-aux-Saints.

As regards the anthropoids Connolly's measurements of fixed brains have shown that the difference in height-length index between chimpanzee, gorilla and orang is not particularly great:

5 male chimpanzees	36·1
6 male gorillas	38·1
13 male orangs	36·5

But these figures do not of course admit of direct comparison with ours, where the technique is quite different. For our own (unfortunately scanty) anthropoid material we have found the following values for internal height-length index or $\left(\frac{\text{internal basion-height} \times 100}{\text{greatest length of skull-cavity}}\right)$, the skull being orientated in the Frankfurt plane:

Chimpanzee	69·8				
Gorilla	73·2,	77·6,	71·7,	66·4,	74·6
Orang	79·0,	83·8,	81·6,	72·9	

All these figures lie within the range of variation for our recent human material! We see how necessary it is also in case of the anthropoids to procure a large body of material for comparison before we can utilise the general shape of the brain for elucidation of the phylogeny. I consider it futile to proceed to speculations respecting the phylogeny on the basis of one or more indices, so long as we have not a large body of material at our disposal and so long as we have not through investigations of correlation secured a more certain basis for correct evaluation of the individual figures. So far as my figures reach, it is permissible to conclude that most prehistoric men, and especially those hitherto best investigated, had a *mesocephalic* brain with a breadth-length index which does not differ materially from that found in many recent races of man, while the height-length index was probably considerably under the mean values for recent races. This increased development in relative height of the brain will accordingly be the real problem so far as regards shape. With respect to the anthropoids, my investigations have confirmed the prevailing assumption of brachycephaly, especially for the orang-outangs, whereas we find to our surprise that the height-length index is not particularly low, provided the technique employed is kept constantly the same. Investigation of a large body of material is here highly necessary, in order to ascertain the mean figures for the different species of anthropoids.

Furthermore we can assert that the difference in general shape of brain between anthropoids and prehistoric and recent races of men is not nearly so great as craniological investigations had hitherto led us to suppose, but it is possible that future investigations will render it justifiable to attach very great importance to the difference that is found to exist. It will contribute very much to a more precise definition of the problems if we consider brain-form and skull-form to some extent separately. It must be due to a confusion between the two—or perhaps rather to deficiency in the figures—when Bolk (3) designates "the primitive races of man," including the Homo Neanderthalensis, as "dolichocephalic," while at the same time the primates are stated to be "generally brachycephalic."

If we consider [illegible] shape of the skull cavity—and otherwise we shall not be able to draw con[illegible]ons—then the primates are certainly brachycephalic as compared with man, but most of the prehistoric human skulls which have hitherto been examined must be deemed to be mesocephalic and not dolichocephalic as stated by Bolk.

V. Endocranial Asymmetry.

It might *a priori* be considered a comparatively simple matter to ascertain which hemisphere was the heavier, or which showed the greater development in length, but nevertheless Hoadley and Pearson quote a large number of papers dealing with this problem without its being possible to say that their excellent survey of the literature makes the matter very much clearer. Boyd, Broca, Topinard, Ogle, Inglessis have published data pointing to "pre-eminence of the left hemisphere." Investigations made by Thurnam, Wagner, Braune and Wilde point in the opposite direction, while Bastian and Reichard leave the question unanswered, the former after having occupied himself therewith for a number of years.

According to Braune and Wilde the greater development of the right cerebral hemisphere is associated with a preponderance of the left cerebellar hemisphere. According to Inglessis the brain is "asymmetrical, but this asymmetry is confined chiefly to the occipital part of the hemispheres, and here the left hemisphere predominates" (Hoadley and Pearson, p. 106), in strong contrast to Broca, who found the preponderance of the left side to be confined to the frontal lobe (cf. the speech centre).

In the following we shall mainly compare our results with those of Hoadley and Pearson, whose technique is the same as ours.

As average of 729 male Egyptian skulls, they find

$$\bar{L}_R = 171{\cdot}0446, \qquad \sigma = 5{\cdot}9551,$$
$$\bar{L}_L = 170{\cdot}0501, \qquad \sigma = 5{\cdot}6908,$$

that is to say, a difference of 0·9945 mm. in the length of the two hemispheres, with preponderance on the right side, a difference which, as the authors show, cannot be due either to "error of measurement" or to "random sampling." "In other words, if we may take our sample of over 700 male Egyptian skulls as representative, the right hemisphere is in the main of greater length, and therefore probably of greater capacity than the left" (p. 111). As stated in the Introduction our figures point in quite the opposite direction:

TABLE VII (*a*).

	Norwegians		Lapps		Maori		Australians	
	♂ $n=81$	♀ $n=76$	♂ $n=63$	♀ $n=55$	♂ $n=16$	♀ $n=14$	♂ $n=13$	♀ $n=10$
L_R	171·94	165·78	167·84	161·42	170·31	167·57	163·30	155·90
L_L	172·54	166·46	168·25	161·71	172·13	168·50	164·15	156·90
$L_L - L_R \pm m_{\text{diff.}}$	0·60 ±0·194	0·68 ±0·154	0·41 ±0·159	0·29 ±0·187	1·82 ±0·385	0·93 ±0·415	0·85 ±0·359	1·00 ±0·379
$L_L - L_R \pm$ probable error	0·60 ±0·131	0·68 ±0·104	0·41 ±0·107	0·29 ±0·126	1·82 ±0·260	0·93 ±0·280	0·85 ±0·242	1·00 ±0·256

In all 8 groups the left half of the cranial cavity is on the average somewhat longer than the right. The differences are from 2·3 times to 7·0 times their probable errors, using the formula

$$0{\cdot}67449\,(\sigma^2_{L_R} + \sigma^2_{L_L} - 2r_{L_R L_L} \cdot \sigma_{L_R} \cdot \sigma_{L_L})^{\frac{1}{2}}.$$

The differences are thus without doubt statistically reliable, not only for the material as a whole, but also for most of the separate groups. The correlations are obviously rather close:

Norwegians ♂: + 0·961, Norwegians ♀: + 0·974. Lapps ♂: + 0·983, Lapps ♀: + 0·957.
Maori ♂: + 0·944, Maori ♀: + 0·937. Australians ♂: + 0·946, Australians ♀: + 0·979.

On measuring a second time the following figures were found, which we give in parentheses in order to avoid confusion:

TABLE VII (*b*).

	Norwegians		Lapps		Maori		Australians	
	♂	♀	♂	♀	♂	♀	♂	♀
L_R	(171·83)	(165·92)	(167·60)	(161·25)	(170·63)	(167·29)	(163·58)	(155·90)
L_L	(172·65)	(166·68)	(168·33)	(161·55)	(172·13)	(168·21)	(163·75)	(156·80)
$L_L - L_R$	(0·82)	(0·76)	(0·73)	(0·30)	(1·50)	(0·92)	(0·17)	(0·90)

We see that the concordance with the first set of measurements is very good. Not only do we find again in all groups the greater length of the left half of the cranial cavity, but there is also an unmistakable correlation between the lateral deviations in the first and the second set of measurements. Only in the small group of female Australians is the difference materially reduced, and here one skull was lacking at the last measurement. The conclusion we can derive from our material as compared with Hoadley and Pearson's therefore is that the asymmetry of the two hemispheres as regards length is *not, as has hitherto been supposed, a characteristic for man as a species, but varies from group to group, probably with preponderance of the left side in most cases**. This view is confirmed by certain other investigations which Hoadley and Pearson do not mention in their survey of the literature. Thus Hrdlička (**10**), who has measured the length of the three cerebral fossae on sectioned skulls, found the sum of the lengths to be greater for the left side: "In seven of the ten series of human adults the average of the sum of the lengths of the three cerebral fossae on the left exceeds that of the right

* It is here assumed that Hoadley and Pearson's technique corresponds exactly to ours. If we omit to light up the cranial cavity and to inspect the posterior point of measurement, this latter will sometimes fall in the furrow for the sinus transversus, which is known to be more developed on the right side. Cf. in this connection what is said later about the correlation between development of the fossa occipitalis sup., depth of the sinus transversus and length of the hemispheres. Cf. also especially Elliot Smith's article (**25**), which also actually concerns Egyptian material.

side by from 1 to 3 mm. There is therefore a clear, though small, excess of fossal length on the left in the mass of human adult crania" (p. 199). This excess on the left side the author finds to be due to the greater length of the left *posterior* fossa (posterior lobe), just as Inglessis found in hardened brains. Cf. Hrdlička, p. 193: "The most striking feature of the above data is the evidence that in all the groups and series the left (posterior) fossa is the longer, which is the reverse of what was observed with the middle and especially with the anterior cavities." The author does not seem to have determined the error of measurement, nor does he add σ and σ_M to his mean figures (the article was published as early as 1907), but the material is comparatively so large (92 skulls) that it must be said to have a certain weight in this connection. A paper recently published by Connolly (7) points in the same direction. On 120 hardened brains the lengths of the two hemispheres were measured by aid of an apparatus designed by the author himself and the accuracy of the results was checked by renewed measurements, partly with change of technique. The material comprised Negroes (37 males, 9 females), Germans (36 males, 10 females), miscellaneous Whites (18 males) and Philippinos (10 males). In 4 of the 6 groups (100 cases) the left hemisphere was from 1 to 3 mm. longer than the right, and only 1 group (10 cases) showed a difference of 1 mm. in the opposite direction. Of the whole material the right hemisphere was longer in 31·7 %, the left was longer in 50·0 %, and both hemispheres equally long in 18·3 % (p. 483). My 41 Eskimos are in accordance with the rule: 18 males $\overline{L}_R = 176{\cdot}06$, $\overline{L}_L = 177{\cdot}00$, thus giving $\overline{L}_L - \overline{L}_R = 0{\cdot}94$ mm.; and 23 females $\overline{L}_R = 167{\cdot}26$, $\overline{L}_L = 168{\cdot}22$, thus giving $\overline{L}_L - \overline{L}_R = 0{\cdot}96$ mm.

All in all, Hrdlička's, Connolly's and my own investigations, together embracing 581 cases, must be said to afford adequate proof of the greater length of the left hemisphere in certain groups of the human population. But it would of course be premature to draw from this the conclusion that the left half of the brain has in the same groups been heavier than the right. In measuring the greatest breadth of the cranial cavity it was strikingly noticeable that the distance from the median line—this latter being determined by the eye, with the longitudinal diameter of foramen magnum and the palatine suture as guiding line—was generally greater on the right than on the left side. Owing to the rough and ready determination of the median plane the figures given below make no claim to particularly great accuracy, but they are recorded here as a first attempt to determine the transverse asymmetry of the cranial cavity.

$\overline{R} - \overline{L}$, maximum distance from the "median line" of the cranial cavity:

Norwegians ♂ (81)	Norwegians ♀ (76)	Lapps ♂ (63)	Lapps ♀ (55)
3·12 mm.	1·76 mm.	2·16 mm.	1·73 mm.
Maori ♂ (16)	Maori ♀ (14)	Australians ♂ (13)	Australians ♀ (10)
1·06 mm.	1·93 mm.	2·31 mm.	2·80 mm.

If we are allowed to assume that the median plane of the skull cavity—as determined above—corresponds to the falx cerebri, or what comes to the same, that the medial surfaces of the cerebral hemispheres are nearly symmetrical to this

plane, then the above-mentioned figures go to show that the right hemisphere has been the broader in these groups.

In order to verify these observations I have measured the corresponding maximum distance from the median plane on horizontal contours through the glabella and upper edge of the orbit. As can easily be ascertained, a contour at this level will go very near to the euryon, and a possible deviation will in any case not be of importance for our problem. Such contours of 166 Lapp skulls (91 male and 75 female) were placed at my disposal by Professor Schreiner. They had been prepared for another purpose after P. and F. Sarasin's system by aid of Martin's Kubus-craniophor and diagraph. The following figures show that the same excess on the right side again appears. The measurements have been taken with an exactitude of $\frac{1}{4}$ mm.

Males	*Females*
Standard error	Standard error
$\overline{R} = 75{\cdot}17 \pm 0{\cdot}32$ mm.	$\overline{R} = 73{\cdot}10 \pm 0{\cdot}36$ mm.
$\overline{L} = 74{\cdot}20 \pm 0{\cdot}28$ mm.	$\overline{L} = 71{\cdot}33 \pm 0{\cdot}32$ mm.
$\overline{R} - \overline{L} = 0{\cdot}97 \pm 0{\cdot}238$ mm.*	$\overline{R} - \overline{L} = 1{\cdot}77 \pm 0{\cdot}290$ mm.*

The distance out to the right side is both for males and females greater than out to the left. In males the difference is $4m_{\text{dif.}}$ and in females $6m_{\text{dif.}}$. If the skulls in which the right side is the broader are designated by +, we get among the 91 males, 52 +, 29 − and 10 zero-deviations; in the 75 females, 51 +, 11 − and 13 zero-deviations. Of the total diagram-material a + deviation to the right is found in 62%, to the left in 24%, and no demonstrable difference in 14%. Technical errors may of course render the above results illusory, but the probability thereof becomes less according as the material is larger and more data can be obtained pointing in the same direction. It is natural to compare my findings with the type contours which have been published in *Biometrika* since 1911. As regards the technique we may refer to Benington and Pearson's article: "Cranial Type Contours," Vol. VIII (1)†. On glabella horizontal sections the parallel No. 7 will give the most adequate expression for the greatest breadth of the skull and a comparison of the distance from the median line on each side affords very good information as to the average transverse asymmetry. Unfortunately some of the series are rather small, and if we examine n which gives the number of skulls, we see that two figures are sometimes given for one series. Obviously a single skull of the series in question has been defective with the result that the right or left side could not be measured. If the series be small, this circumstance will render it useless for our purpose. I have therefore in Table VIII omitted the series consisting of less than 10 skulls, in cases where the right or left side was not measured on one or more of the skulls.

[* If the correlations between R and L are anything like those shown on p. 111, these probable errors are curiously high. ED.

† The reader should examine this carefully as much turns on the manner in which the median lines are found in the horizontal and transverse contours. Von Bonin's technique is not that of the Biometric Laboratory. ED.]

Altogether the table embraces 42 series. These show a + deviation to the right in 32 cases and to the left in 10 cases, and on adding our own investigations of the dimensions of the cranial cavity the numbers will be 40 and 10 respectively, as we here disregard the craniograms of our Lapp material, since this has in part been included in the 40. Moreover, as already mentioned, these craniograms also proved to be in accordance with the rule. Of course, the distribution in some of the smaller groups will be fortuitous, while *technical errors* undoubtedly may be responsible for some of the minor deviations (Congo Batetelu, Eskimo, Morant's and Woo's Egyptian crania, Bantam-Batavia and Mittel Java, as well as Negro crania from Teita Hills), but for the larger series with average difference in breadth of more than 1 mm. it is very improbable that this source of error has played any important rôle. This assumption can easily be confirmed by comparison with the corresponding "transverse vertical contours" which are likewise shown in Table VIII. It is seen that on these latter contours + deviation in 27 cases goes to the right, in 14 cases to the left, while in 1 case the deviation is 0. If we confine the examination to the groups in which the number of skulls equals or exceeds 20 and where the excess in breadth on one side is equal to or more than 1 mm. on transverse contours, we find that in 14 out of 15 cases the excess is on the right side and only in 1 case on the left, which altogether is a good proof that the technical errors have not been so great that the rule (the preponderating + deviation on the right side) loses its general validity, even though we cannot venture to attach any decisive weight in this question to the individual investigations. It must therefore be said to be of minor importance that 37 Basque crania or a single group of 31 Eskimo skulls show + deviation to the left, but it is of greater interest to note that out of 9 English groups there is only 1 (Hythe female crania) which on horizontal contours diverges from the rule. Or, if we prefer also to take into account the transverse contours, we find that of 18 deviations 16 go to the right and only 2 to the left (male Anglo-Saxons, transverse contours, and female Hythe crania, horizontal contours). Of Egyptian material we have 6 series which without exception, both on transverse and on horizontal contours, are in accordance with the rule, whereas the Oriental material (Burmese, Tibetan A and B, Nepalese, Hindu) strikingly often show a + deviation to the left. I thus obtain confirmation for my earlier expressed assumption that cerebral asymmetry is probably a racially determined characteristic (associated with the quantitative proportion of left-handedness in the population?), although for hitherto investigated European material it can be regarded as a general rule that the left hemisphere is the longer and the right side of the cranial cavity is the broader*.

In this connection I must mention a paper published by Woo (34) which to some extent confirms my findings. Woo has, like Hoadley and Pearson, investigated a very large body of Egyptian material—887 male skulls from the 26th to the 30th Dynasty—with a special view to asymmetry conditions. His technique differs

* The plausible conjecture that the greater breadth of one side represented a compensation for the greater length of the other has not been confirmed by our material. The correlation [$L_L - L_R$ and $B_R - B_L$] is practically equal to 0 and is in any case without statistical significance. ($r = -0{\cdot}060 \pm 0{\cdot}055$.)

TABLE VIII.

Author	Biometrika Vol.		Horizontal Contours ♂ n	R	L	R–L	Horizontal Contours ♀ n	R	L	R–L	Transverse Contours ♂ R	L	R–L	Transverse Contours ♀ R	L	R–L
Benington, C. ...	VIII	Congo cr., *a*	50	67·3	66·7	0·6	—	—	—	—	66·6	65·4	1·2	—	—	—
		„ *b*	20	68·9	66·3	2·6	—	—	—	—	66·8	66·5	0·3	—	—	—
		„ Batetelu	41	68·7	68·8	−0·1	—	—	—	—	67·7	66·4	1·3	—	—	—
		Egyptian cr., XXVI to XXX D.	100	68·2	67·5	0·7	—	—	—	—	66·5	65·5	1·0	—	—	—
		Eskimo cr.	31	66·0	66·3	−0·3	—	—	—	—	66·5	67·1	−0·6	—	—	—
		Guanche cr.	14	70·0	68·8	1·2	—	—	—	—	67·7	66·0	1·7	—	—	—
		English cr., 17th century ...	100	69·8	69·4	0·4	—	—	—	—	68·6	68·5	0·1	—	—	—
Thomson, E. Y. ...	XI	Moriori cr.	33	69·1	68·3	0·8	21	67·1	65·8	1·3	68·6	67·2	1·4	65·7	64·4	1·3
Tildesley, M. L. ...	XIII	Burmese cr., A	44 (43)	70·7	70·2	0·5	39	68·5	66·6	1·9	69·7	70·0	−0·3	66·9	66·5	0·4
		„ B	7	69·6	67·6	2·0	17	67·3	65·7	1·6	67·4	68·3	−0·9	65·8	65·8	0·0
		„ C	8	69·0	68·0	1·0	18	64·1	62·9	1·2	67·4	67·8	−0·4	63·6	62·7	0·9
Morant, G. M. ...	XIV	Tibetan cr., A	17*	68·7	70·5	−1·8	—	—	—	—	68·4	69·8	−1·4	—	—	—
		„ B	15	68·1	70·6	−2·5	—	—	—	—	69·2	68·8	0·4	—	—	—
Morant, G. M. ...	XVI	Oriental cr., Tibetan A	35	68·8	70·2	−1·4	—	—	—	—	67·3	67·6	−0·3	—	—	—
		„ Nepalese	46	67·0	66·1	0·9	—	—	—	—	63·2	63·7	−0·5	—	—	—
		„ Hindu	10	63·3	64·2	−0·9	—	—	—	—	62·2	63·4	−1·2	—	—	—
	XVIII	Anglo-Saxon cr.	37	70·5	67·8	2·7	39	66·0	65·0	1·0	68·8	69·2	−0·4	66·2	66·1	0·1
Morant, G. M. ...	XVII	Egyptian cr. I D.	(31)	66·9	66·6	0·3	—	—	—	—	66·1	64·9	1·2	—	—	—
Hooke, B. G. E. ...	XVIII	17th-century Londoners	70 (69)	70·5	69·2	1·3	65 (64)	66·4	65·3	1·1	69·5	67·4	2·1	65·4	63·5	1·9
Stoessiger, B. N. ...	XIX	Badarian cr. [Egypt]	34	65·9	63·1	2·8	21	64·7	62·6	2·1	63·6	62·6	1·0	63·7	61·1	2·6
Morant, G. M. ...	XXI	Basque cr.	37 (36)	69·2	70·6	−1·4	—	—	—	—	68·9	69·7	−0·8	—	—	—
Woo, T. L.	XXII	Sedment cr. [Egypt, IX D.] ...	38	67·4	67·1	0·3	30	65·5	64·5	1·0	66·3	65·8	0·5	63·9	63·2	0·7
Bonin, G. von ...	XXIII	Dajaks	30	68·8	67·0	1·8	—	—	—	—	66·2	68·0	−1·8	—	—	—
		Bantam-Batavia	54	69·7	69·5	0·2	—	—	—	—	69·9	67·3	2·6	—	—	—
		Mittel-Java	30	70·2	69·9	0·3	—	—	—	—	72·1	68·0	4·1	—	—	—
		Tagalen	29	68·7	67·6	1·1	—	—	—	—	67·7	66·6	1·1	—	—	—
		Aëtas	30	70·5	71·8	−1·3	—	—	—	—	71·5	69·5	2·0	—	—	—
Bonin, G. von ...	XXIII	Easter Islander cr.	12	65·3	66·5	−1·2	—	—	—	—	63·4	68·1	−4·7†	—	—	—
Morant, G. M. ...	XXIII	Spitalfields cr.	116 (115)	71·7	70·2	1·5	22 (21)	67·1	65·9	1·2	70·6	70·0	0·6	66·1	65·7	0·4
Kitson, E.	XXIII	Negro cr., Teita hills	46	64·7	64·3	0·4	49	62·8	62·4	0·4	63·7	63·8	−0·1	61·4	61·6	−0·2
Stoessiger and Morant	XXIV	Hythe cr. (English)	112	73·6	72·4	1·2	87	69·3	70·0	−0·7	72·4	72·3	0·1	69·5	68·8	0·7

* These 17 Tibetan A are part of the 35 Tibetan A below.

† "The fact that the Easter Islanders show an asymmetry of the skull reverse to that seen in the Anglo-Saxons may be noticed in passing" (p. 265).

very much from mine, as only external measurements were taken and then chiefly confined to the individual bones. While paying full regard to possible sources of error, the author concludes that a preponderance on the right side is more frequent than on the left. Out of 25 measurements 16 show a + deviation to the right and, as appears from his tables, this excess is most pronounced for the os parietale and the os frontale. On the other hand it is noteworthy that the occipital arc from the lambda to the asterion "is very significantly greater on the left," and although the two other occipital measurements give contrary results, it therefore seems rather premature when the author on the basis of the excess on the right side in the frontal and parietal region draws the conclusion that the right hemisphere is more strongly developed (p. 332)*. To my former conclusion I shall here merely add that the question of a general preponderance of the right or of the left hemisphere can hardly be decided by cranial measurements, and in carrying out direct weighing—which will certainly be adopted very largely also in the future—the anthropological character of the material ought not to be entirely disregarded. It would be of great interest to examine the larger lobes more systematically than has hitherto been the case, and judging from my material I should expect to find *right-sided preponderance of the parietal and left-sided preponderance of the occipital lobe, here again with certain reservations for anthropological reasons.*

Relation between fossa occipitalis sup., sinus transversus dexter et sinister and length of the hemispheres.

So far as can be seen, Elliot Smith was the first to point out that the upper posterior fossa, which so clearly marks off the caudal pole of the posterior lobe of the brain, is in general deeper and more fully formed on the left side than on the right (25). It was natural to connect this finding with the main outflow of the venous blood through the right sinus transversus. As mentioned in most text-books on anatomy and as can easily be ascertained on inspecting a series of crania, the furrow is more strongly developed on the right side than on the left—Le Double (15)—while the foramen jugulare also shows distinct difference on the two sides. I must agree with Elliot Smith that it is natural to suppose—but only as a provisional hypothesis—that the greater development of the left posterior lobe is *the cause* of the main outflow of the venous blood being to the right. And the next step in the process of deduction really follows of itself: the greater development of the left posterior lobe is in some way or other connected with the predominating use of the right hand, so that by examining the fossa occipitalis superior and the sinus transversus in prehistoric man and in anthropoids we can obtain interesting information as to the phylogenetic development of "right-handedness." Elliot Smith has later reverted to this question on several occasions, for instance in an article on "Right and Left-handedness in Primitive Men" (26), where it is stated, *inter alia*, that the Pithecanthropus skull-cap and the London skull both show inverse occipital asymmetry, for which reason both of them are

[* It is in the frontal and parietal regions on the *left* side that functional centres have been authoritatively placed and are asserted to give predominance in weight and size to that side. Ed.]

thought to have belonged to left-handed individuals. Here, however, we already meet with the first difficulties, since it is found that the sinus transversus in both of these cases runs to the right. For there at once arises two questions of fundamental importance in this connection: (1) Does the deeper and more marked fossa occipitalis sup. sinistra give expression for a greater development of the left posterior lobe? (2) Can there be demonstrated any correlation between the craniological characters, the development of the fossa occipitalis sup. and the depth of the sinus transversus?* To the first question Elliot Smith originally thought he could give an affirmative answer. For he found on hardened brains that the area striata, i.e. "the area of cortex containing Gennari's stria," extended far more widely over the outer convex surface of the hemisphere on the left side, and he interpreted this finding as a sign that the left area strata was much more diffused, but later in 1907 he adds—"which subsequent investigation has shown to be erroneous" (p. 574), and he describes in this connection an attempt to measure the area striata directly on 10 brains. As the results did not confirm the original assumption, he advances the theory that it is the greater development of certain other cortical regions which causes the right area striata to fold itself largely on the medial surface of the hemisphere, and he mentions specially as a possibility Flechsig's large parietal association centre, which is stated to be much larger on the right side. This, too, is of course merely a conjecture, but it seems to be worth mentioning on the background of my own and Woo's investigations respecting cranial asymmetry. That the left region of vision is not more developed than the right has subsequently been established by Cohn and Papez (6). Measurements of the fissura calcarina on 100 hardened brains—and according to Campbell (5) the length of the fissura calcarina gives expression for the extension of the region of vision—gave 8·8 cm. for the right side and 7·9 cm. for the left. In no less than 72 % the furrow was longer on the right side, as against 10 % longer on the left. These measurements are unaccompanied by information as to the influence of error of measurement, but the authors verify their results by direct planimetric measurements, which led to practically the same result. I shall pass from this question without bringing into the discussion the difficult problem of "left-handedness." I refer to the above-mentioned paper by Cohn and Papez, and above all to Woo and Pearson's (32): "Dextrality and Sinistrality of Hand and Eye," where the latter authors' reserved attitude to the question finds expression in the accompanying motto: "Keine Erklärung ist aber doch besser als eine oder mehrere irrthümliche."

Further Elliot Smith believed he could answer affirmatively the second question. He states that in a series of Egyptian skulls (which we may certainly be permitted to assume to have been large in number) about 80 % show marked asymmetry, with left-sided preponderance of the fossa occipitalis sup. (25, p. 576), and at the same place it is said that in the cases where there was distinct divergence the main

* A third and not less important question is whether the inverse occipital asymmetry really is associated with left-handedness. This question Elliot Smith claims to be able to answer in the affirmative and he refers to a comparative investigation made by Wood-Jones in 1925 (35) respecting the length of the bones of the upper arm.

direction of the transverse sinus was to the left. No statistical *proof* of this relationship is furnished, however, and Tildesley, who took up the same question in 1920, arrived at the following negative result: "One has to conclude therefore that in this series—127 Burmese skulls—there is no significant association between outer occipital asymmetry and the direction in which the main occipital sinus is turned when its course is changed from longitudinal to lateral" (27, p. 257). Meanwhile Tildesley points out that this *external* examination of the squama occipitalis is not decisive for the question discussed by Elliot Smith, which thus still remains unanswered, at any rate from a statistical standpoint. It is on this point that our material is calculated to cast some light.

The characters here concerned are in their manifestation of a quantitative nature, although it is not inconceivable that the underlying biological causative factors are alternative (dextrality or sinistrality). Meanwhile it is practically impossible quantitatively to measure the size of the posterior fossa or of the sinus transversus. On the first examination of the material I have noted the depth (size) of the sinus transversus in three categories, *r*, *l* and ?, and on the second investigation noted the development of the posterior fossa in the same manner. The cases where the one groove was deeper, but the other of greater breadth, are assigned to the ?-group. As regards the statistical treatment of the data, I have arranged my material into 9 cell-systems and to these applied "the root-mean-square contingency," the formula for which is

$$\phi^2 = \frac{1}{n} \Sigma \frac{(n_{pq} - v_{pq})^2}{v_{pq}},$$

where n_{pq} stands for observed and v_{pq} for theoretically expected frequencies. This dimension is compared with

$$\phi^2 \text{ for zero-correlation} = \frac{c-1}{n} \pm \cdot 67449 \sqrt{\frac{2c}{n}},$$

where c is the number of "cells," and further, for the sake of completeness, C_2 (coefficient of mean-square contingency) has been reckoned out from the formula

$$C_2 = \sqrt{\frac{\phi^2}{1+\phi^2}}.$$

I have found it justifiable, on the basis of our earlier investigations of asymmetry, which revealed the same conformity to rule in all groups, to deal with the material collectively, without regard to sex or race.

We will first examine the relation between *length of hemisphere* and *size of fossa occipitalis sup.*

Greater Length of Hemisphere

Larger fossa occipitalis	Right	Equal	Left	Totals
Right	24	14	12	50
Equal	19	45	45	109
Left	9	53	104	166
Totals	52	112	161	325

From the vertical marginal totals it is seen that our material shows the same regularity as regards the fossa occipitalis sup. as Elliot Smith's Egyptian series: 51·1% are larger on the left side, against 15·4% on the right side. The group "equal" forms as much as 33·5%, which shows that our estimation has been very cautious. We see that the group "left-left" is by far the most numerous and the group "right-right" considerably more numerous than the dissimilar combinations.

$\phi^2 = 0{\cdot}1846$, as against $0{\cdot}0246 \pm 0{\cdot}0088$ for zero contingency and $C_2 = 0{\cdot}3947$ (without "class index correction"). We can conclude from these figures that the greater development of the left posterior fossa is a very important, *possibly* the only, cause of the greater length of the left hemisphere in our material.

Relation between fossa occipitalis sup. and sinus transversus:

Greater Groove

Larger fossa occipitalis	Right	Equal	Left	Totals
Right	12	15	23	50
Equal	73	24	12	109
Left	135	25	6	166
Totals	220	64	41	325

From the horizontal marginal totals it can be reckoned out that the transverse sinus in 67·7% of the skulls goes preponderatingly to the right, as against 12·6% to the left, which accords well with earlier observations made by Le Double (68·5% right, 14·5% left), Elliot Smith and Tildesley (70·1% right, 19·7% left). Further we find that 158 skulls are in accordance with Elliot Smith's law—right-left and left-right combinations, while only 18 skulls form an exception.

$\phi^2 = 0{\cdot}1973$, which on comparison with the earlier mentioned value for zero contingency furnishes a good proof of the validity of the law. ($C_2 = 0{\cdot}4059$.)

The relation between the sinus transversus and length of the cranial cavity is seen from the following figures:

Greater Length of Hemisphere

Greater Groove	Right	Equal	Left	Totals
Right	12	70	140	222
Equal	11	33	20	64
Left	30	10	2	42
Totals	53	113	162	328

It is seen that the dissimilar combinations are by far the most numerous, the contingency is even higher than before:

$\phi^2 = 0{\cdot}3810$, while ϕ^2 for zero contingency is about as before.

$C_2 = 0{\cdot}5786$.

We thus find a high contingency between length of hemisphere and main direction of the sinus transversus, assuming that length of cranial cavity forms an adequate expression for length of hemisphere. We have thereby succeeded in furnishing statistical proof of the validity of Elliot Smith's law, and by a method which—as far as regards one of the characters—is independent of all subjective judgment. It is reasonable to *suppose*, as does Elliot Smith, that there exists a comparatively simple causality relation between the characters, but we cannot *a priori* preclude the possibility that the correlation is due to other circumstances, for example, the influence of some common factor (position of the fetus in the uterus?). It may possibly be supposed that such (hypothetical) factors may be the cause of the correlation not being absolute. In our last table there are two cases where the left hemisphere has the greater length while at the same time the flow of blood goes mainly to the left, and in twelve cases the situation is the opposite. In these cases it may be that factors of otherwise secondary importance have played a decisive part.

Especially it must be noted that our figures tell nothing as to the relation to left-handedness. In this respect a systematic investigation of length of hemisphere in connection with the size of the sinus transversus and the length of the humerus on each side would undoubtedly be illuminating.

The results of our investigations as to the asymmetry of the skull—and, indirectly, of the brain—may be briefly summed up as follows:

1. Left hemisphere longer than right. (+ deviation to left in 49·4 %, + deviation to right in 16·2 %.)

2. Right hemisphere broader than left. (+ deviation to right in 67·1 %, *certain* + deviation to left only in 2·2 %.)

3. Left fossa occipitalis sup. more developed than right. (Left fossa greater in 51·1 %, right fossa greater in 15·4 %.)

4. Right sinus transversus larger than left. (Right in 67·7 %, left in 12·6 %.)

Hereunto may be added that, according to Elliot Smith as well as to Cohn and Papez, the right area striata is greater than the left, while at the same time we may remember that Hoadley and Pearson's investigations of length of hemisphere in their Egyptian material give results different from ours. The most important problems remaining in this field accordingly are the following:

(1) Will direct measurements on fixed brains confirm our findings as to the greater breadth of the right hemisphere, and if so, what is the cause of the greater breadth (parietal lobe)?

(2) How are we to explain the far greater development of the left fossa occipitalis sup. when at the same time the area striata is less developed on the left side?

(3) How can we explain the cases in which the sinus transversus deviates from the rule?

VI. DIMENSIONS OF CRANIAL CAVITY AND PREDICTION OF CAPACITY.

It would lead too far to give a detailed account of the many attempts that have been made to set up formulae for calculation of capacity. In this respect we may refer to Martin's text-book, while a recently published paper by Neuert (19) also gives a survey of the literature on the subject. It will be found that in the main two principles have been followed. The investigators have *either* (1), as do Broca and Manouvrier, begun by taking the product of the chief cranial dimensions (L, B, H), occasionally with measurement of arcs and circumferences, and then introduced a number of corrections in order to reduce the error to the smallest possible amount, *or else* (2) they have, like Lee and Pearson, computed the correlation between the directly observed capacity values and the product of the three main diameters (eventually taking arcs and circumferences) and on the basis of this correlation established the best possible regression formula. This latter mode of procedure (No. 2) might be designated the *empirical-statistical* method and there can hardly be any doubt but that it is scientifically better founded than the former, which is of *a priori* nature*. The English-speaking, German and Scandinavian anthropologists have by degrees united in adopting the English method, and the formulae most frequently employed are the following†:

I. Lee's interracial formula, computed from 8 different races.

$$\text{Male } C = 0{\cdot}000\,370 \,.\, L \,.\, B \,.\, OH + 321{\cdot}16 \text{ cm.}^3$$
$$\text{Female } C = 0{\cdot}000\,375 \,.\, L \,.\, B \,.\, OH + 296{\cdot}40 \text{ cm.}^3$$

II. Pearson's interracial formula.

$$\text{Male } C = 0{\cdot}000\,365 \,.\, L \,.\, B \,.\, OH + 359{\cdot}34 \text{ cm.}^3$$
$$\text{Female } C = 0{\cdot}000\,375 \,.\, L \,.\, B \,.\, OH + 296{\cdot}40 \text{ cm.}^3$$

In case the ear-height is not known, other formulae come into consideration, as to which we may refer to Martin's text-book. The same applies to the *intra-racial formulae*, which are of course to be preferred in many cases. The importance of having a good interracial formula is, however, obvious. Sometimes the race is unknown or doubtful, as, for example, in dealing with prehistoric crania, while in other cases an adequate intraracial formula is lacking. Meanwhile it is maintained in several quarters that these mathematically determined interracial formulae are not

* Instead of linear measurements Neuert adopts measurement of plane surfaces (cephalograms), taken in different planes of the skull, sometimes several surfaces in the same plane. The advantage of this method is that peculiarities in the *shape* of the cranium find better expression. The method seems rather complicated and can hardly be said to be fully elaborated statistically, but judging from Neuert's paper the results are very good. For X-ray examination of living individuals it undoubtedly has a future before it.

[† As to the attribution of these formulae, see my Note following this Memoir. ED.]

particularly satisfactory, and the reason for this may be found in four factors: (1) Influence of bone thickness, (2) development of the sinus frontalis, (3) peculiarities in the shape of the skull and (4) fortuitous circumstances in connection with the composition of the material forming the basis for the formulae, both with respect to selection of races and to the numerical proportion of the individual races. In the paper which has already several times been mentioned Hoadley and Pearson (9) therefore investigate the correlation between the product of the three chief internal diameters, L_i, B_i and H_i (by H_i is understood here the internal basion-height, perpendicular on the Frankfurt horizontal) and the capacity. The latter was determined "by packing the skull tightly with mustard seed and then weighing." It was found that the correlation had increased from 0·82081 (between capacity and external measurements) to 0·89586 (between capacity and internal measurements). The possibility of obtaining a better formula was therefore obvious. With a view to rendering the method more serviceable in practice the authors tried to determine L_i, B_i and H_i from L, B and H, but this attempt failed owing to the slight correlation between thickness of bone and external measurement. The authors therefore conclude by stating that "we must either be contented with the degree of accuracy provided by the external diametral product, or if we wish to improve on that accuracy we must practise the rather difficult technique of internal measurement" (p. 93). Owing to the great technical difficulties the authors express doubt as to the utility of the new mode of procedure. As my new instrument renders the technique simpler and more time-saving, while at the same time the accuracy of measurement becomes greater, fresh interest is created in the importance of internal measurements in determination of capacity. And there can be no doubt that the correlation between the internal measurements and the capacity has also in our material been considerably increased as compared with the external measurements, but I have nevertheless refrained from this mode of procedure, as our material comprises too few races—two of the races especially (Maori and Australians) being too badly represented—to form the basis for an empirical-statistical inter-racial formula. Such a formula would, in fact, not have a long life. According as more races were investigated it would be necessary to make small changes in its different factors. I have therefore selected another mode of procedure, which no doubt is theoretically to be compared with the *a priori* method, but which I think rests upon a better foundation than all previous attempts in that direction, and which in practice will give better results than Lee's and Pearson's formulae. If we disregard the individual peculiarities of the cranial cavity and compare it to a regularly shaped space, then it is most natural to think of the shape of an egg, and the stereometrically known body that will first come into consideration is *the ellipsoid*, the simple formula for which,

$$C = \frac{L.Br.H}{6} . 3{\cdot}14159,$$

it should be possible to apply to the three main diameters of the cranial cavity As height of the ellipsoid we have chosen the internal ear-height (external ear-bregma height minus thickness of the os parietale) instead of the internal

basion-height or internal opisthion-height, which owing to the greatly varying clivus-angle would give less satisfactory results. And instead of the maximum breadth of the cavity alone I have for the breadth of the ellipsoid used the mean of greatest breadth of cavity + maximum frontal breadth. If we inspect a series of skulls from one or more race in the norma verticalis, we soon become aware of the greatly varied tapering of the anterior pole of the skull. It is mainly this circumstance that forms the basis for Sergi's classification of skull types into Pentagoids, Ellipsoids, Rhomboids, etc.; cf. contours on pp. 688—689 of Martin's text-book. By introducing the *maximum frontal breadth* in addition to the internal measurements we therefore hope to eliminate or diminish three of the previously mentioned sources of error. The ideal procedure would of course be to use the *internal frontal breadth*, but this dimension is very difficult to obtain and, besides, the thickness of the bone at this place is of very little importance.

The formula we employ is therefore the following:

$$C = 0{\cdot}0005236 . L_i . \left(\frac{B_1 + B_2}{2}\right) . H_i,$$

where $L_i = \frac{L_L + L_R}{2}$, B_1 = greatest breadth of the cranial cavity, B_2 = greatest frontal breadth, H_i = internal ear-bregma-height and and $0{\cdot}0005236 = \frac{3{\cdot}14159}{6 \times 1000}$.

We shall first see how this formula suits our material without an additive term. The directly observed capacities in the following table have been found by means of the mustard-seed method and measuring-glass. Ranke's bronze skull has invariably been employed for checking the packing.

TABLE IX.

	Norwegians		Lapps		Maori		Australians	
	♂ $n=81$	♀ $n=76$	♂ $n=63$	♀ $n=55$	♂ $n=16$	♀ $n=14$	♂ $n=13$	♀ $n=10$
Directly observed C	1456 ±14·3	1293 ±11·4	1473 ±13·8	1301 ±11·3	1418 ±20·2	1327 ±23·3	1278 ±21·9	1113 ±17·3
Computed C... ...	1234	1093	1247	1123	1195	1109	1072	931
Observed-computed C	222	200	226	178	223	218	206	182

The best possible intraracial formulae are accordingly obtained by adding to the expression $0{\cdot}0005236 . L_i \left(\frac{B_1 + B_2}{2}\right) . H_i$ as additive term the values given in the lowest column. The next step in the investigation should be to apply these formulae so as to ascertain the individual divergencies. Here we lack, however, intraracial

formulae for comparison, and therefore we proceed at once to use the interracial formula. It is seen that the additive terms show remarkably little variation. This is of course a necessary condition for a good interracial formula when the other factor should at the same time be kept constant. On weighting the additive terms with their respective n-values we get the following interracial formula, which we will at once compare with Lee's and Pearson's*:

$$\text{For } ♂ \text{ skulls } C = 0{\cdot}0005236\, L_i \cdot \left(\frac{B_1 + B_2}{2}\right) \cdot H_i + 222 \text{ cm.}^3$$

$$\text{For } ♀ \text{ skulls } C = 0{\cdot}0005236\, L_i \cdot \left(\frac{B_1 + B_2}{2}\right) \cdot H_i + 192 \text{ cm.}^3$$

The following Table X gives us an opportunity of comparing the mean figures for the three methods with the direct measurements.

TABLE X. *Mean figures.*

	Mustard Seed	Lee	Pearson	Ellipsoid formula	Mustard Seed Lee	Mustard Seed Pearson	Mustard Seed Ellipsoid formula
Norwegians ♂	1456	1443	1463	1456	+13	− 7	0
" ♀	1293	1279	1279	1285	+14	+14	+ 8
Lapps ♂	1473	1419	1444	1469	+54	+29	+ 4
" ♀	1301	1294	1294	1315	+ 7	+ 7	−14
Maori ♂	1418	1428	1450	1417	−10	−32	+ 1
" ♀	1327	1300	1301	1301	+27	+26	+26
Australians ♂	1278	1333	1357	1294	−55	−79	−16
" ♀	1113	1178	1178	1123	−65	−65	−10

It is seen that in six groups the ellipsoid formula gives better results than Lee's and Pearson's methods and in one group (Lapps ♀) the difference is only 7 cm.^{3} in the opposite direction. In the group Maori ♀ the difference is insignificant. Lee's and Pearson's formulae suit very badly for our Australian skulls. Somewhat better results would here be obtained by following Wacker's instruction to add 50 cm.^{3} for thick-walled skulls (Martin, p. 647—really intended it only to apply to Lee's method). The results would, however, still be worse than with the ellipsoid method, and moreover Lee's and Pearson's formulae would in that way lose their inter-racial character.

Table XI shows the mean individual divergencies, without regard to plus or minus sign.

It is seen that the ellipsoid formula brings us on the average from 5 to 53 cm.^{3} nearer to the directly observed values.

* In the additive terms we entirely neglect the decimals, which are without practical importance. The exact figures would be 222·35 and 192·66 cm^3. After the figures 222 and 192 had been employed one of the Australian skulls was rejected as being defective, which explains the slight divergency as regards ♀.

TABLE XI.

Average Individual Divergency, without regard to sign.

	Lee	Pearson	Ellipsoid formula
Norwegians ♂	54	55	47
" ♀	45	45	33
Lapps ♂	69	56	35
" ♀	40	40	35
Maori ♂	28	41	23
" ♀	42	42	37
Australians ♂	69	85	32
" ♀	65	65	22

Here again it is the Australian material that gives the worst results by Lee's and Pearson's methods. A deduction of 50 cm.^{3} renders the figures somewhat more favourable, but they still continue to be worse than by my new method, the average divergency being 45 or 53 cm.^{3} for males and 29 cm.^{3} for females as against respectively 32 and 22 cm.^{3} by the ellipsoid formula.

An investigation of the standard deviation of the divergencies with zero as mean has given the following results:

TABLE XII.

Standard Deviation of Divergencies from directly observed C.

Race	Mustard Seed Lee	Mustard Seed Pearson	Mustard Seed Ellipsoid
Norwegians ♂	70·1 ± 5·5	72·8 ± 5·7	58·0 ± 4·6
" ♀	55·5 ± 4·5	55·5 ± 4·5	41·0 ± 3·3
Lapps ♂	82·5 ± 7·3	69·1 ± 6·2	44·3 ± 3·9
" ♀	53·2 ± 5·1	53·2 ± 5·1	41·3 ± 3·9
Maori ♂	33·8 ± 6·0	46·2 ± 8·2	31·5 ± 5·6
" ♀	62·2 ± 11·8	62·2 ± 11·8	46·8 ± 8·8
Australians ♂	75·1 ± 14·7	94·3 ± 18·5	37·6 ± 7·4
" ♀	72·3 ± 16·2	72·3 ± 16·2	28·6 ± 6·4

Now it might naturally be objected against the comparison here made that the ellipsoid formula has been applied to a material which had served as basis for the establishment of the same formula. As the additive term is the only factor which varies, the question to be considered is the influence of that factor on the comparison. There are two circumstances which reduce the weight of the objection. Firstly, the fact that our material includes two such widely divergent races as Lapps and Australians, wherefore it is probable that the formula will give correspondingly good results also for races which as regards shape, capacity and thickness of skull lie between these extremes. And, secondly, the objection will not apply at all to the

two groups, female Lapps and female Maori, where the mean figures for the capacities computed by the ellipsoid formula are not better than those obtained by Lee's and Pearson's methods. *It is seen from Table XI that even in these cases the average individual divergency is reduced, although only by* 5 *cm.*³ Although the advantages of the new method can hardly be doubted, yet for control I have applied it to an *entirely new material.* For this purpose a series of Eskimo skulls, 18 males and 23 females, likewise belonging to the Anatomical Institute, was selected. On account of the peculiar wedge-shaped tapering of the parietalia this material should provide an especially good test. From the figures given below it appears that here also the ellipsoid formula gives much better results than the other two methods:

Eskimos.

Mean values	*Male*	*Female*
Directly observed C ...	1526 cm.³	1335 cm.³
Lee	1480 „	1318 „
Pearson	1502 „	1318 „
Ellipsoid formula ...	1522 „	1336 „
Average divergency		
Lee	75 cm.³	48 cm.³
Pearson	69 „	48 „
Ellipsoid formula ...	40 „	35 „

The standard deviation of the divergencies from the directly observed capacities:

For males: $\sigma_{\text{Mustard Seed: Lee}}\ 75{\cdot}5 \pm 12{\cdot}6$,

$\sigma_{\text{Mustard Seed: Pearson}}\ 77{\cdot}0 \pm 12{\cdot}8$, $\sigma_{\text{Mustard Seed: Ellipsoid}}\ 51{\cdot}3 \pm 8{\cdot}55$.

For females: $\sigma_{\text{Mustard Seed: Lee}}\ 53{\cdot}5 \pm 7{\cdot}9$, $\sigma_{\text{Mustard Seed: Ellipsoid}}\ 45{\cdot}8 \pm 6{\cdot}8$.

TABLE XIII.

	Norwegians ♂, $n=81$					Norwegians ♀, $n=76$					Lapps ♂, $n=63$				
External measurements	*M*	±	σ	±	*v*	*M*	±	σ	±	*v*	*M*	±	σ	±	*v*
1. Maximum length	187·70	0·65	5·86	0·46	3·12	179·28	0·67	5·81	0·47	3·24	179·17	0·86	6·79	0·60	3·79
2. Maximum breadth	142·34	0·56	5·03	0·40	3·53	135·37	0·45	3·94	0·32	2·91	146·52	0·63	5·00	0·45	3·41
3. Basion-bregma height	132·31	0·73	6·55	0·51	4·95	125·34	0·54	4·75	0·39	3·79	129·05	0·57	4·52	0·40	3·50
4. (*a*) Ear-bregma height	113·28	0·55	4·99	0·39	4·41	108·16	0·42	3·66	0·30	3·38	113·02	0·45	3·55	0·32	3·14
(*b*) 4 (*a*) − thickness of parietal bone (6 (*a*)) ...	107·96	0·52	4·72	0·37	4·37	103·78	0·39	3·39	0·27	3·27	108·71	0·45	3·60	0·32	3·31
5. (*a*) Maximum frontal breadth	118·16	0·60	5·41	0·43	4·58	113·77	0·42	3·67	0·30	3·23	120·59	0·56	4·42	0·39	3·67
(*b*) (5 (*a*)+14)/2	126·10⁵	0·52	4·71	0·37	3·72	120·99	0·36	3·10	0·25	2·56	130·33	0·51	4·02	0·36	3·08
6. (*a*) Thickness of parietal bone	5·33	0·17	1·57	0·12	[29·45]	4·38	0·13	1·17	0·09	[26·71]	4·30	0·15	1·23	0·11	[28·60]
(*b*) 3 − 15	5·25	0·20	1·80	0·14	[34·29]	4·32	0·18	1·57	0·13	[36·34]	4·16	0·19	1·52	0·14	[36·54]
(*c*) ½ (2 − 14)	4·15	0·09	0·86	0·07	[20·72]	3·57	0·10	0·86	0·08	[24·09]	3·24	0·11	0·84	0·07	[25·92]
7. (*a*) Thickness of occipital bone, *R*	4·30	0·13	1·16	0·09	[26·98]	4·10	0·14	1·21	0·10	[29·51]	3·12	0·14	1·12	0·10	[35·90]
(*b*) Thickness of occipital bone, *L*	4·10	0·13	1·21	0·10	[29·51]	3·86	0·13	1·13	0·09	[29·27]	2·90	0·13	1·03	0·09	[35·52]
(*c*) (7 (*a*)+7 (*b*))/2	4·21	0·11	1·00	0·08	[23·75]	3·98	0·13	1·12	0·09	[28·28]	3·01	0·13	1·04	0·09	[34·55]
8. 1 − 13 (*c*) − 7 (*c*)	11·22	0·33	2·97	0·23	[26·47]	9·21	0·27	2·32	0·19	[25·36]	8·11	0·40	3·21	0·29	[39·58]
9. Breadth-length index (2 × 100)/1	75·88	0·33	2·95	0·23	3·89	75·56	0·31	2·69	0·22	3·56	81·90	0·52	4·16	0·37	5·08
10. Height-length index (3 × 100)/1	70·46	0·40	3·64	0·29	5·17	69·96	0·34	2·95	0·24	4·22	72·05	0·39	3·09	0·28	4·29
11. Height-breadth index (3 × 100)/2	93·08	0·58	5·26	0·41	5·65	92·67	0·52	4·53	0·37	4·89	88·19	0·55	4·34	0·39	4·92
Internal measurements															
13. (*a*) Maximum sagittal length of skull cavity, *R*	171·94	0·68	6·14	0·48	3·59	165·78	0·67	5·88	0·48	3·55	167·84	0·80	6·37	0·57	3·80
(*b*) Maximum sagittal length of skull cavity, *L*	172·54	0·70	6·32	0·50	3·66	166·46	0·68	5·92	0·48	3·56	168·25	0·85	6·74	0·60	4·01
(*c*) (*R*+*L*)/2	172·24	0·69	6·24	0·49	3·62	166·12	0·68	5·90	0·48	3·55	168·05	0·83	6·56	0·58	3·90
14. Maximum breadth of skull cavity	134·05	0·49	4·44	0·35	3·31	128·22	0·45	3·90	0·32	3·04	140·05	0·61	4·88	0·43	3·48
15. Internal basion height	127·07	0·72	6·48	0·51	5·10	121·03	0·52	4·49	0·36	3·71	124·88	0·58	4·64	0·41	3·72
16. Internal opisthion height	129·28	0·67	6·02	0·47	4·66	126·09	0·58	5·03	0·41	3·99	127·60	0·73	5·76	0·51	4·51
17. Internal breadth-length index (14 × 100)/13 (*c*)	77·56	0·37	3·29	0·26	4·24	77·24	0·29	2·57	0·21	3·33	83·46	0·50	3·96	0·35	4·74
18. Internal height-length index (15 × 100)/13 (*c*) ...	73·89	0·45	4·02	0·32	5·44	73·84	0·39	3·43	0·28	4·65	74·45	0·45	3·57	0·32	4·80
19. Internal height-breadth index (15 × 100)/14 ...	94·82	0·64	5·79	0·45	6·11	94·48	0·54	4·69	0·38	4·96	89·40	0·56	4·46	0·40	4·99
20. Capacity (mustard seed)	1456·0	14·3	128·6	10·1	8·8	1293·0	11·4	99·8	8·1	7·7	1473·0	13·8	109·3	9·7	7·4
21. Capacity (Lee)	1443·0	10·3	92·5	7·3	6·4	1279·0	8·5	74·4	6·0	5·8	1419·0	10·0	79·2	7·1	5·6
22. Capacity (Pearson)	1463·0	10·1	90·9	7·1	6·2	1279·0	8·5	74·4	6·0	5·8	1444·0	9·8	77·5	6·9	5·4
23. Capacity (ellipsoid formula)	1456·0	12·2	110·1	8·7	7·6	1285·0	9·0	78·5	6·4	6·1	1469·0	12·4	98·6	8·8	6·7

TABLE XIII (*continued*).

	Lapps ♀, $n=55$					Maori ♂, $n=16$					Maori ♀, $n=14$				
External measurements	*M*	±	σ	±	v	*M*	±	σ	±	v	*M*	±	σ	±	v
1. Maximum length	170·96	0·71	5·23	0·50	3·06	186·81	1·04	4·14	0·73	2·22	180·00	0·95	3·55	0·67	1·97
2. Maximum breadth	142·05	0·59	4·39	0·42	3·09	137·88	1·15	4·58	0·81	3·32	132·64	0·97	3·64	0·69	2·74
3. Basion-bregma height	124·05	0·58	4·29	0·41	3·46	136·63	1·15	4·61	0·82	3·37	130·21	1·03	3·84	0·73	2·95
4. (*a*) Ear-bregma height	109·51	0·52	3·88	0·37	3·54	116·06	0·78	3·11	0·55	2·68	112·07	1·28	4·79	0·91	4·27
(*b*) 4 (*a*) – thickness of parietal bone (6 (*a*)) ...	105·53	0·50	3·68	0·35	3·49	109·19	0·73	2·92	0·52	2·67	106·29	1·30	4·87	0·92	4·58
5. (*a*) Maximum frontal breadth	116·27	0·58	4·27	0·41	3·67	112·63	0·98	3·90	0·69	3·46	109·50	0·87	3·25	0·61	2·97
(*b*) (5 (*a*)+14)/2	125·74*	0·49	3·61	0·34	2·87	121·78*	0·92	3·69	0·65	3·03	118·39*	0·82	3·08	0·58	2·60
6. (*a*) Thickness of parietal bone	3·98	0·14	1·07	0·10	[26·88]	6·88	0·35	1·41	0·25	[20·49]	5·79	0·32	1·20	0·23	[20·73]
(*b*) 3 – 15	3·44*	0·23	1·67	0·16	[48·54]	6·19	0·53	2·10	0·37	[33·93]	4·64	0·47	1·77	0·33	[38·15]
(*c*) ½ (2 – 14)	3·41	0·12	0·86	0·08	[25·07]	3·28	0·14	0·56	0·10	[17·07]	3·04	0·20	0·75	0·14	[24·67]
7. (*a*) Thickness of occipital bone, *R*	3·07	0·15	1·12	0·11	[36·48]	5·06	0·29	1·14	0·20	[22·53]	4·50	0·26	0·98	0·19	[21·78]
(*b*) Thickness of occipital bone, *L*	3·02	0·15	1·09	0·10	[36·09]	4·72	0·22	0·86	0·15	[18·22]	4·57	0·28	1·05	0·20	[22·98]
(*c*) (7 (*a*)+7 (*b*))/2	3·06*	0·14	1·02	0·10	[33·33]	4·89	0·24	0·94	0·17	[19·22]	4·54	0·28	1·05	0·20	[23·13]
8. 1 – 13 (*c*) – 7 (*c*)	6·33*	0·34	2·51	0·24	[39·65]	10·71	0·52	2·09	0·37	[19·51]	7·43	0·45	1·70	0·32	[22·88]
9. Breadth-length index (2×100)/1	83·17	0·49	3·65	0·35	4·39	73·87	0·73	2·90	0·51	3·93	74·07	0·66	2·48	0·47	3·35
10. Height-length index (3×100)/1	72·62	0·39	2·89	0·28	3·99	73·15	0·46	1·85	0·33	2·53	72·35	0·50	1·86	0·35	2·57
11. Height-breadth index (3×100)/2	87·39	0·47	3·49	0·33	3·99	99·23	1·22	4·86	0·86	4·90	98·26	1·13	4·23	0·80	4·30
Internal measurements															
13. (*a*) Maximum sagittal length of skull cavity, *R*	161·42	0·64	4·77	0·45	2·96	170·31	0·99	3·95	0·70	2·32	167·57	1·19	4·44	0·84	2·65
(*b*) Maximum sagittal length of skull cavity, *L*	161·71	0·62	4·61	0·44	2·85	172·13	1·14	4·55	0·80	2·64	168·50	1·14	4·26	0·81	2·53
(*c*) (*R*+*L*)/2	161·57	0·63	4·65	0·44	2·88	171·21	1·06	4·25	0·75	2·48	168·04	1·17	4·37	0·83	2·60
14. Maximum breadth of skull cavity	135·24	0·56	4·18	0·40	3·09	131·31	1·12	4·46	0·79	3·40	126·57	1·03	3·87	0·73	3·06
15. Internal basion height	120·62	0·55	4·10	0·39	3·39	130·44	0·94	3·74	0·66	2·87	125·57	1·05	3·93	0·74	3·13
16. Internal opisthion height	124·29	0·61	4·55	0·43	3·66	126·81	1·28	5·15	0·91	4·06	129·77	1·18	4·42	0·84	3·41
17. Internal breadth-length index (14×100)/13 (*c*)	83·77	0·46	3·43	0·33	4·09	77·12	0·71	2·85	0·50	3·70	75·36	0·64	2·39	0·45	3·17
18. Internal height-length index (15×100)/13 (*c*) ...	74·72	0·42	3·14	0·30	4·20	75·94	0·73	2·90	0·51	3·82	74·74	0·52	1·96	0·37	2·62
19. Internal height-breadth index (15×100)/14 ...	89·37	0·51	3·75	0·36	4·20	99·28	1·22	4·87	0·86	4·91	99·30	1·14	4·26	0·81	4·29
20. Capacity (mustard seed)	1301·0	11·3	84·0	8·0	6·5	1418·0	20·2	80·9	14·3	5·7	1327·0	23·3	87·2	16·5	6·6
21. Capacity (Lee)	1294·0	8·7	64·4	6·1	5·0	1428·0	17·4	69·4	12·3	4·9	1300·0	19·0	71·0	13·4	5·5
22. Capacity (Pearson)	1294·0	8·7	64·4	6·1	5·0	1450·0	17·0	68·0	12·0	4·7	1300·0	19·0	71·0	13·4	5·5
23. Capacity (ellipsoid formula)	1315·0	10·4	77·0	7·3	5·9	1417·0	17·7	70·8	12·5	5·0	1301·0	23·0	86·1	16·3	6·6

[* These values are not in accordance with the statements in the first column. ? replace by 125·755, 3·43, 3·045, 6·845, 121·97 and 118·035 respectively. Ed.]

[* These values are not in accordance with the statements in the first column. ?replace by 125·755, 3·43, 3·045, 6·345, 121·97 and 118·035 respectively. Ed.]

TABLE XIII (*continued*).

	Australians ♂, $n=13$					Australians ♀, $n=10$				
External measurements	M	±	σ	±	v	M	±	σ	±	v
1. Maximum length	183·30	1·21	4·36	0·85	2·38	173·40	1·69	5·33	1·19	3·07
2. Maximum breadth	130·85	1·27	4·57	0·90	3·49	124·60	1·10	3·47	0·78	2·78
3. Basion-bregma height	135·92	1·26	4·55	0·89	3·35	130·70	0·95	2·99	0·67	2·29
4. (*a*) Ear-bregma height	113·85	1·26	4·55	0·89	4·00	107·80	1·04	3·28	0·73	3·04
(*b*) 4 (*a*) – thickness of parietal bone (6 (*a*)) ...	106·46	0·76	2·74	0·54	2·57	102·10	1·08	3·42	0·76	3·35
5. (*a*) Maximum frontal breadth	111·23	1·35	4·85	0·95	4·36	104·90	0·91	2·88	0·64	2·75
(*b*) (5 (*a*)+14)/2	117·31	1·25	4·49	0·88	3·83	111·25	0·88	2·78	0·62	2·50
6. (*a*) Thickness of parietal bone	7·39	0·56	2·02	0·40	[27·33]	5·70	0·32	1·00	0·22	[17·54]
(*b*) 3 – 15	7·15	0·64	2·32	0·45	[32·45]	4·80	0·77	2·44	0·55	[50·83]
(*c*) ½ (2 – 14)	3·73	0·30	1·07	0·21	[28·69]	3·50	0·22	0·71	0·16	[20·29]
7. (*a*) Thickness of occipital bone, *R*	5·08	0·44	1·57	0·31	[30·91]	5·40	0·35	1·11	0·25	[20·56]
(*b*) Thickness of occipital bone, *L*	5·31	0·45	1·61	0·32	[30·32]	5·45	0·08	0·26†	0·06	[4·77]
(*c*) (7 (*a*)+7 (*b*))/2	5·19	0·39	1·42	0·28	[27·36]	5·42	0·31	1·01	0·23	[18·63]
8. 1 – 13 (*c*) – 7 (*c*)	14·35	0·83	2·99	0·59	[20·84]	11·58	0·74	2·35	0·53	[20·29]
9. Breadth-length index (2 × 100)/1	71·38	0·51	1·85	0·36	2·59	71·92	0·83	2·61	0·58	3·63
10. Height-length index (3 × 100)/1	74·21	0·64	2·31	0·45	3·11	75·40	0·60	1·91	0·43	2·53
11. Height-breadth index (3 × 100)/2	103·99	1·22	4·39	0·86	4·22	105·04	0·82	2·59	0·58	2·47
Internal measurements										
13. (*a*) Maximum sagittal length of skull cavity, *R*	163·30	1·10	3·97	0·78	2·43	155·90	1·60	5·09	1·14	3·26
(*b*) Maximum sagittal length of skull cavity, *L*	164·15	1·08	3·88	0·76	2·36	156·90	1·76	5·58	1·25	3·56
(*c*) (*R*+*L*)/2	163·77*	1·07	3·85	0·76	2·35	156·40	1·69	5·35	1·20	3·42
14. Maximum breadth of skull cavity	123·38	1·18	4·25	0·83	3·44	117·60	0·96	3·04	0·68	2·59
15. Internal basion height	128·77	1·42	5·12	1·00	3·98	125·90	1·34	4·25	0·95	3·38
16. Internal opisthion height	129·08	1·16	4·18	0·82	3·24	126·30	1·15	3·64	0·81	2·88
17. Internal breadth-length index (14 × 100)/13 (*c*)	75·35	0·62	2·24	0·44	2·97	75·27	0·90	2·85	0·64	3·79
18. Internal height-length index (15 × 100)/13 (*c*) ...	78·67	0·96	3·45	0·68	4·39	80·61	1·21	3·83	0·86	4·75
19. Internal height-breadth index (15 × 100)/14 ...	104·48	1·48	5·33	1·05	5·10	107·05	0·71	2·24	0·50	2·09
20. Capacity (mustard seed)	1278·0	21·9	79·1	15·5	6·2	1113·0	17·3	54·8	12·3	4·9
21. Capacity (Lee)	1333·0	22·8	82·3	16·1	6·4	1178·0	22·3	70·5	15·8	6·3
22. Capacity (Pearson)	1307·0	22·4	80·9	15·9	6·2	1178·0	22·3	70·5	15·8	6·3
23. Capacity (ellipsoid formula)	1294·0	19·2	69·1	13·6	5·3	1123·0	20·8	65·9	14·7	5·9

* ?163·72[5].

[† This value and the results on either side dependent on it are remarkable, and if correct show how little dependence can be placed on series of 10. Ed.]

TABLE XIII (*continued*).

	Eskimos ♂, $n=18$					Eskimos ♀, $n=23$				
External measurements	*M*	±	σ	±	*v*	*M*	±	σ	±	*v*
1. Maximum length	190·44	1·17	4·97	0·83	2·61	179·65	0·95	4·58	0·68	2·70
2. Maximum breadth	137·56	1·06	4·49	0·75	3·26	132·43	0·96	4·60	0·68	3·47
3. Basion-bregma height	139·39	1·13	4·79	0·80	3·44	133·52	0·86	4·13	0·61	3·09
4. (*a*) Ear-bregma height	119·56	0·86	3·64	0·61	3·04	114·52	0·65	3·10	0·46	2·71
(*b*) 4 (*a*) − thickness of parietal bone (6 (*a*)) ...	114·92	0·81	3·42	0·57	2·98	110·30	0·73	3·48	0·51	3·16
5. (*a*) Maximum frontal breadth	113·94	0·83	3·54	0·59	3·11	110·04	0·91	4·36	0·64	3·96
(*b*) (5 (*a*)+14)/2	122·36	0·84	3·57	0·59	2·92	117·93	0·93	4·44	0·65	3·76
6. (*a*) Thickness of parietal bone	4·64	0·26	1·10	0·18	[23·71]	4·22	0·24	1·14	0·17	[27·01]
(*b*) 3 − 15	5·78	0·51	2·15	0·36	[37·20]	5·65*	0·51	2·46	0·36	[43·54]
(*c*) ½ (2 − 14)	3·33*	0·30	1·29	0·21	[38·74]	3·30	0·25	1·20	0·18	[36·36]
7. (*a*) Thickness of occipital bone, *R*	5·67	0·48	2·04	0·34	[35·98]	5·15	0·29	1·39	0·20	[26·99]
(*b*) Thickness of occipital bone, *L*	5·08	0·40	1·68	0·28	[33·07]	5·20	0·28	1·32	0·19	[25·38]
(*c*) (7 (*a*)+7 (*b*))/2	5·38	0·44	1·86	0·31	[34·57]	5·17	0·25	1·22	0·18	[23·60]
8. 1 − 13 (*c*) − 7 (*c*)	8·38*	0·60	2·56	0·43	[30·55]	6·74	0·52	2·51	0·37	[37·24]
9. Breadth-length index (2 × 100)/1	72·30	0·84	3·57	0·59	4·94	73·75	0·58	2·78	0·41	3·77
10. Height-length index (3 × 100)/1	73·23	0·66	2·79	0·46	3·81	74·36	0·54	2·59	0·38	3·48
11. Height-breadth index (3 × 100)/2	101·44	1·07	4·54	0·76	4·48	100·93	0·88	4·22	0·62	4·18
Internal measurements										
13. (*a*) Maximum sagittal length of skull cavity, *R*	176·06	1·10	4·67	0·78	2·65	167·26	0·94	4·50	0·66	2·69
(*b*) Maximum sagittal length of skull cavity, *L*	177·00	1·02	4·32	0·72	2·44	168·22	0·99	4·75	0·70	2·82
(*c*) (*R*+*L*)/2	176·53	1·08	4·59	0·76	2·60	167·74	0·94	4·50	0·66	2·68
14. Maximum breadth of skull cavity	130·78	1·13	4·81	0·80	3·68	125·83	1·07	5·14	0·76	4·08
15. Internal basion height	133·61	1·10	4·68	0·78	3·50	127·91	0·79	3·81	0·56	2·98
16. Internal opisthion height	131·39	1·23	5·20	0·87	3·96	127·43	1·07	5·11	0·75	4·01
17. Internal breadth-length index (14 × 100)/13 (*c*)	74·15	0·87	3·69	0·61	4·98	75·04	0·59	2·81	0·41	3·74
18. Internal height-length index (15 × 100)/13 (*c*) ...	75·73	0·70	2·97	0·49	3·92	76·31	0·65	3·10	0·46	4·06
19. Internal height-breadth index (15 × 100)/14 ...	102·31	1·21	5·15	0·86	5·03	101·83	1·04	4·97	0·73	4·88
20. Capacity (mustard seed)	1526·0	18·3	77·5	12·9	5·08	1335·0	19·1	91·8	13·5	6·88
21. Capacity (Lee)	1480·0	14·3	60·5	10·1	4·09	1318·0	12·2	58·4	8·6	4·43
22. Capacity (Pearson)	1502·0	14·0	59·6	9·9	3·97	1318·0	12·2	58·4	8·6	4·43
23. Capacity (ellipsoid formula)	1522·0	14·4	61·1	10·2	4·01	1336·0	17·1	81·8	12·1	6·13

[* ? Read instead of these respectively 3·39, 8·535 and 5·61. Ed.]

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[* ? Read instead of these respectively 3·39, 8·535 and 5·61. ED.]

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NOTE ON SECTION VI OF DR K. WAGNER'S MEMOIR.

By Karl Pearson.

1. I find it difficult to follow the reasoning of a portion of Dr Wagner's most valuable paper. But before I enter into any criticism, it is needful to remind the reader of one or two points which all of us are apt to overlook. There are two types of formulae respectively termed by me many years ago, *intraracial* and *interracial*. If we wish to predict a character like the capacity of the skull which is laborious to measure, or it may be impossible, from more easily ascertained lengths, such as external diameters, then we require to know the most probable value and the standard error, which determines the range of variation about this most probable value. The statistical method is definite—provided we make certain assumptions—i.e. we measure the dependent character desired and the independent characters in as many individuals of the same race as are available and construct a regression surface, and theory then gives us the probable value and the standard variation about that value of the character desired in terms of the independent variables. On the assumption that cranial characters approximately follow what are termed normal distributions, the resulting regression surface is planar, and the standard variation depends on the multiple correlation coefficient. The result thus obtained enables us to predict from the given set of independent variables the desired character in any number of the same race with a given likelihood of accuracy. Such a construction formula is termed an *intraracial* formula.

As long as the hypothesis of normality is sensibly true, and therefore our formula linear, the mean of any desired character in a number n of crania of the same race will be obtained with the same accuracy by inserting in the formula the means of the independent variables.

If we now take the means of the desired character in a number of races and the corresponding means of the independent characters, these means will follow normal distributions, if the individual characters do; and we construct in the same manner a planar regression surface for ascertaining the mean desired character in any new race from the means of its independent variables. Such a regression surface is termed an *interracial* formula.

The process is perfectly straightforward if laborious from both the measurement and computing sides, once the hypothesis of normal distribution of cranial characters is accepted; and, as theory demonstrates, no other planar formula with the *same independent* variables can give as good, much less better, results. The formulae, whether obtained by the theory of multiple regression or by the method of least squares, are identical.

In the above manner intraracial reconstruction formulae were worked out for five races, in 1901, by Dr Lee and myself*. We worked them out for various combinations of the Length, Breadth and Height of the skull, including the product of these diameters. These intraracial formulae gave by no means bad results and average formulae from our best material, Bavarian, Aino and Naqada, gave us general formulae for the two sexes (*loc. cit.* p. 243), namely,

$$\left.\begin{aligned}(\delta)\ C &= \cdot 000337\,LBH + 406\cdot 01\\ (\varphi)\ C &= \cdot 000400\,LBH + 206.60\end{aligned}\right\} \quad \ldots\ldots\ldots\

It must be remembered that the mean capacity of a number of crania can only be obtained approximately from these formulae by inserting the mean values of LBH,—for the mean value of the product (LBH) is not the product of the mean values of L, B, and H.

Now let us turn to interracial formulae. Few investigators have measured enough crania of a large number of sufficiently diverse races to obtain a good interracial formula, whether it be a linear or product formula. The use of a large number of racial capacities measured by different observers may be most misleading; not only may the technique be different, but variations due to personal equation are excessive. The older French measurements cannot be very safely pooled with the German, and the older English measurements with either*. In the course of the Royal Society memoir of 1901 Dr Lee and I reached by the method of least squares interracial formulae (*loc. cit.* p. 247) for male and female series. These are as follows:

$$\left.\begin{array}{l}(\delta)\ C = \cdot 000365\, \bar{L} \times \bar{B} \times \bar{H} + 359{\cdot}34 \\ (\varphi)\ C = \cdot 000375\, \bar{L} \times \bar{B} \times \bar{H} + 296{\cdot}40\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

where $\bar{H}$ is again the mean auricular height measured perpendicular to the Frankfort Plane.

These results were obtained from the mean capacities, and the mean diameters of 9 or 10 races; for this reason they cannot be applied to an individual skull of any special race. They are formulae for *mean* values: cf. what has been said on p. 133.

I have referred to the difficulty of getting capacity measured in the same manner by different observers and techniques, and accordingly in a footnote Dr Lee and I gave the male formula with three doubtful series omitted, i.e.

$$(\delta)\ C = \cdot 000370 L \times B \times H + 321{\cdot}16 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

To my surprise Dr Wagner breaks up Dr Lee's and my joint work, and provides two sets of formulae. He attributes to Dr Lee (ii) and to me (iii) for males and the second of (ii) for females. It has the appearance of our having provided two sets of different and partly discordant formulae! Why he should separate out a result in a footnote to a joint memoir as peculiarly *my* fundamental formula for racial capacities I am puzzled to say!

On the same page (*loc. cit.* p. 247) are given the interracial formulae

$$\left.\begin{array}{l}(\delta)\ C = \cdot 000266 \bar{L} \times \bar{B} \times \bar{H}' + 524{\cdot}6 \\ (\varphi)\ C = \cdot 000156 \bar{L} \times \bar{B} \times \bar{H}' + 812{\cdot}0\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)},$$

where $\bar{H}'$ is the mean height of the skull measured from the basion to the point of the skull *vertically above* the basion when orientated to the Frankfort horizontal plane.

Now the reader will note that neither H (= OH of later notation) nor H' are measured to the *bregma*.

It would have been theoretically more reasonable to take instead of L the length L', the *gerade Länge* of the *Frankfurter Verständigung*, for then L', B and H (or H') would have been mutually at right angles and have represented a true physical right six-face, and the product would have been its actual volume. This was not possible in our investigations, for the length projection L' was not provided by most craniologists. Thus LBH, or LBH', still less $\frac{1}{2}(L_R + L_L) \times B_i \times H_i$, cannot be looked upon as the volumes of right six-faces, closely embracing the skull. Such products are merely arbitrary functions of characters of the skull, and there is no theoretical justification for multiplying them by $\frac{4\pi}{3} \times \frac{1}{8}$ as a factor, on the basis of the volume of an ellipsoid. The skull may resemble an ellipsoid, but if so its capacity must be measured by the product of three mutually rectangular axes, and not by quantities like LBH which are not mutually rectangular.

Of course we may start from any function f of cranial external or internal variables, and its fitness will depend on how intense the correlation of C and f may be. That is the sole test of

* On these points see the paper by M. A. Lewenz and K. Pearson, "On the measurement of Internal Capacity from Cranial Circumferences," *Biometrika*, Vol. III. pp. 366—397, 1904. Cf. especially p. 377.

whether one f is or is not better than a second. Dealing with intraracial data for Egyptians, Series E, Pearson and Hoadley found a correlation of C and $L_i B_i H_i$ of ·8959; they accordingly judged that the product $(L_i B_i H_i) = P_i$ was better than $(LBH) = P$ which gave a correlation of ·8208 only. They did not go further than providing the intraracial reconstruction formula for Ancient Egyptians*:

$$(\sigma) \quad C = \cdot 0004372 P_i + 169 \cdot 57 \pm 34/\sqrt{n} \quad \ldots\ldots\ldots\

as an intraracial formula for the Norwegian skull; nor does he make clear how he changes from the L_i, B_i and H_i for an individual skull to a product $\bar{L}_i\,\bar{B}_i\,\bar{H}_i$ of racial means, for the mean "ellipsoid" of many skulls would not be the "ellipsoid" of mean axes. Even Dr Wagner's theoretical justification for using the ellipsoidal coefficient, ·0005236, falls to the ground when he uses $\frac{1}{2}(B_f+B_i)$ for his intermediate axis, for if we give an ovoid form to the skull by using a maximum breadth and a lesser breadth nearer to one end, then the coefficient for the volume will not take the ellipsoidal value. All experience shows that the coefficient of the product term does vary from sex to sex and from race to race. Indeed, as I shall show, a better prediction formulae than Dr Wagner gets for his four races can be at once obtained by considering both coefficients in $C=AP_i+B$ as unknown and determined by least square or correlational methods.

But there is a point which must be regarded in all these product prediction formulae. We have seen that roughly the characters measured on the skull follow normal distributions. Accordingly the regression formula for C on any *linear* function of L, B, H will be planar, but it by no means follows that the regression formula for C' on a product will be linear. No theoretical demonstration of this has so far been given*, and at any rate in the case of the product of the three circumferential arcs Lewenz and Pearson have shown by graphing the product to the capacity that for a wide range of races it is very unlikely that linearity exists†.

2. Let us look a little into the arithmetic of the formulae cited.

The actual result of Dr Wagner's discussion is that he applies for males the intraracial formula for Norwegians as an interracial formula, namely:

$$C=\cdot0005236\bar{L}_i\times\tfrac{1}{2}(\bar{B}_f+\bar{B}_i)\times\bar{H}_i+222 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{vii}^{\text{bis}}),$$

where the bars above the variables denote racial means, and

$\bar{L}_i=\frac{1}{2}(L_R+L_L)$=mean of internal lengths=the same quantity as used by Hoadley and Pearson in 1929;

$\bar{B}_i$=maximum internal breadth, agreeing with Hoadley and Pearson's measurements;

B_f=maximum external frontal breadth. This was a character not originally selected for measurement in the Biometric Laboratory, for its terminals frequently fall on very complicated sutures, so that they are very doubtful and subject to large personal equation. Dr Wagner does not state how he dealt with much indented sutures. H_i with Dr Wagner is the distance from bregma to the auricular axis less the thickness of the parietal bone. This latter should be in the *direction of the external measurement* and *at the bregma*, but our author takes it on the side wall of the parietal bone, thus assuming that there is such a general character as the "thickness of the parietal bone." The Hoadley-Pearson H_i is the internal basion height, perpendicular to the Frankfort Plane. We will call this H_i' for the time being and we have the following values for two races, the Norwegian and Egyptians of the 20th to 30th Dynasties:

Race	$\bar{L}_i$	$\frac{1}{2}(\bar{B}_f+\bar{B}_i)$	$\bar{H}_i$	$\bar{P}_i$	$\bar{B}_i$	$\bar{H}_i'$	$\bar{P}_i'$
Norwegians	172·28	126·51	107·96	235,3003·617	134·05	127.07	293,4571·607
Egyptians	170·44	124·03‡	107·97§	228,2450·515	132·14	128·85	290,1952·175

* I am not overlooking the fact that if variations from the mean be substituted for L, B and H, then LBH can be expanded as a linear function to a *first* approximation; but in this case we might as well start with a linear function and accordingly discard any of the advantages supposed to arise from the use of the product function.

† See *Biometrika*, Vol. III. 1904, p. 387.

‡ $\bar{B}_f=115·92$.

§ $\bar{H}_i$=Bregma auricular height − parietal thickness=113·43 − 5·46=107·97.

Now let us apply to these products the interracial male formula (vii) of Wagner (intraracial for the Norwegians) and the intraracial formula of Pearson and Hoadley for the Egyptians. We have:

	Norwegians	Egyptians
Wagner ...	1454 (−2)	1417 (−23)
Pearson-Hoadley	1453 (−3)	1438 (− 2)
Observed	1456	1440

These results suggest three points, namely that:

(i) There would not appear to be any advantage obtainable by introducing $\overline{B}_f$, a measure undoubtedly difficult to determine accurately and destroying the hypothesis that we are taking *internal* measurements.

(ii) There is no special advantage in using the ellipsoidal coefficient ·0005236.

(iii) We are as likely to get as good a result from the Pearson-Hoadley form of product P_i' as from Dr Wagner's P_i.

We will accordingly discard the unsatisfactory $\overline{B}_f$, and the still more unsatisfactory $\overline{H}_i$ with its measurement to the bregma and its inadequate bregmatic thickness, and see whether the Pearson-Hoadley P_i' does or does not give as good results for an interracial formula as Wagner's formula. He had the advantage of measuring the brain-box internally for five races in both sexes, using in all 191 male and 182 female skulls. All his series were short, some of them terribly short, in six of the ten cases under 25 crania! We took the largest series in the Biometric Laboratory, 729 male adult crania, because our purpose was to test how far internal measurements gave a better result than external measurements, freeing ourselves as far as possible from the errors of random sampling. And we found they did—an improvement resulted of about 25 °/₀. Now Dr Wagner's great merit is to have increased our supply of data by measuring internally five races to which we can add a sixth, our Egyptians series. I have not the boldness to suggest an interracial formula based on four, or even six races, but I do wish to show that a formula better fitting those six races can be found by the usual processes of statistical theory, than by "guessing" A and then averaging B. Before doing so I should like to point out that for racial comparisons, as far as cranial capacity is concerned, but little stress can be laid on racial estimates of capacity which differ from the observed by 25 to 50 cm.³ Thus consider four of Dr Wagner's shorter series and compare them with previously determined capacities for longer series:

Dr Wagner's Series		Race	Previous Values	
Capacity	Size of Sample		Capacity	Size of Sample
1418	16	Maori ...	1476	43
1278	13	Australians	1295	134
1526	18	Eskimos ...	1559	34
1473	63	Lapps ...	1522	66

(References are given on p. 143.)

It is obvious from these results that even if Dr Wagner's interracial formula gave exactly in these four races his observed capacities, those capacities would not agree within 58, 18, 33, and

49 cm.^{3} with the values obtained from larger samples. This may be due to the size of the samples, to differences in technique, or to the influence of local races; but, clearly, any interracial prediction formulae coming within 25 cm.^{3} to 50 cm.^{3} of each other may easily be equally valid, if we are determining a "racial" character, not seeking a short cut to find the mean capacity of a series of crania.

I propose now to consider in the first place three prediction formulae for cranial capacities on the basis of the six races for which internal cranial measurements have been taken, namely:

(*a*) Dr Wagner's formula.

(*b*) The "best" formula that could be obtained from Dr Wagner's data, by the use of correlational theory.

(*c*) The Pearson-Hoadley form of product, using all the available data.

We need the following values:

Race	$\bar{L}_i$	$\frac{1}{2}(\bar{B}_f+\bar{B}_i)$	$\bar{H}_i$	$\bar{P}_i$	$\bar{B}_i$	$\bar{H}_i'$	$\bar{P}_i'$
Norwegians	172·28	126·51	107·96	235,3003·617	134·05	127·07	293,4571·607
Lapps ...	168·05	130·33	108·71	238,0961·691	140·05	124·88	293,9101·064
Maori ...	171·21	121·78	109·19	237,6606·455	131·31	130·44	293,2497·960
Australians	163·77	117·31	106·46	204,5294·477	123·38	128·77	260,1919·229
Eskimos ...	176·53	122·36	114·92	248,2296·255	130·78	133·61	308,4599·744
Egyptians E	170·44	124·03*	107·97	228,2450·515	132·14	128·85	290,1952·175

From these we obtain for (*b*) by the method of correlation,

$$C = \cdot 00056241178\,\bar{P}_i + 136{\cdot}345 \quad \ldots\ldots (b),$$

and for (*c*) by the same method,

$$C = \cdot 00051495732\,\bar{P}_i' - 61{\cdot}083 \quad \ldots\ldots (c),$$

while (*a*) is as before,

$$C = \cdot 0005236\,\bar{P}_i + 222 \text{ cm.}^3 \quad \ldots\ldots (a).$$

From these formulae results the following table:

Race	Observed Capacity	Wagner's (*a*)	"Best" $\bar{P}_i$ Formula (*b*)	Pearson's $\bar{P}_i'$ Formula (*c*)
Norwegians ...	1456	1454 (− 2)	1460 (+ 4)	1450 (− 6)
Lapps ...	1473	1469 (− 4)	1475 (+ 2)	1452 (−21)
Maori ...	1418	1414 (− 4)	1417 (− 1)	1449 (+31)
Australians ...	1278	1293 (+15)	1287 (+ 9)	1279 (+ 1)
Eskimos ...	1526	1522 (− 4)	1532 (+ 6)	1527 (+ 1)
Egyptians E	1440	1417 (−23)	1420 (−20)	1433 (− 7)
Mean Root Square Residual		11·59	9·47	15·75

Equation (*b*) shows that using Wagner's value for P_i the ellipsoidal coefficient does not give the best values, nor does Wagner's value of the constant term. We improve the root mean square residual by about 9·4 per cent. by using a correct statistical procedure. We see further as a result of the above table that the Pearson-Hoadley P_i' gives a worse result than Dr Wagner's,—at least when we deal with the above six races. This is a misfortune, for its $\bar{L}_i$, $\bar{B}_i$ and $\bar{H}_i'$ are so much

* $\bar{B}_f$ for these Egyptians = 115·92 mm.

easier of determination than using both $\overline{B}_f$ and $\overline{B}_i$, and the complicated and theoretically very doubtful $\overline{H}_i$. But it is clear that using the internal vertical basion height H_i, so easy to determine, does not provide a formula so good, judged by the square root mean square residual, as Dr Wagner's. There are points to be noted however in the formula (*c*). It gives the highest and lowest race capacities excellently, far better than (*a*) or (*b*); it fails peculiarly with the Maori capacity, but here it must be noted that Dr Wagner's measure of capacity is much below that of Scott, indeed by almost 50 cm.^{3}. I defer a closer discussion of this point till we turn to external diametral product formulae. Meanwhile I will try to reach a formula better than Dr Wagner's in which all the quantities to be measured can be obtained in a fairly simple manner, and which avoids his $\overline{B}_f$ and $\overline{H}_i$.

We have seen so far that the vertical basion height, while not bad for a component of an interracial formula, is not as good as Dr Wagner's $\overline{H}_i$. Let us see what the vertical opisthion height $\overline{H}_i''$ will lead us to.

The table of needful measurements is the following:

Race	$\overline{L}_i$	$\overline{B}_i$	$\overline{H}_i''$	P_i''	Predicted Value	Deviation from Observed
Norwegians ...	172·28	134·05	129·28	298,5609·964	1476·5	+20
Lapps ...	168·05	140·05	127·60	300,3117·359	1485	+12
Maori ...	171·21	131·31	124·29	279,4236·212	1383	−35
Australians ...	163·77	123·38	129·08	260,8183·071	1291	+13
Eskimos ...	176·53	130·79	131·39	303,3347·507	1500	−26
Egyptians E...	170·44	132·14	130·65	294,2491·670	1455	+15

Obtaining the "best" formula by the method of least squares, we have:

$$C = \cdot00049042125\ \overline{L}_i \overline{B}_i \overline{H}_i'' + 12{\cdot}310 \quad \text{.....(viii).}$$

Substituting the products P_i'' in the formula, we obtain the predictions given in the sixth column of the above table, which provide the deviations from the observed values given in the seventh column. These have a root mean square residual of 21·75 cm.^{3}, not a very serious resulting error when we think of the amounts by which one sample of a race differs from a second, but worse than the formula which uses the vertical basion height, and very considerably worse than the "best" formula using Wagner's P_i. At first sight it would appear as if nothing had been obtained by investigating the results of using the vertical basion and vertical opisthion internal heights. But a careful examination of the two sets of deviations shows that if we use formula (*c*) in round numbers seven times and formula (viii) five times, i.e. weight them with $\frac{7}{12}$ and $\frac{5}{12}$ respectively, the large deviations will all be reduced to insignificant proportions. We reach by this procedure the formula:

$$\bar{C} = (\cdot00030039177\ H_i' + \cdot00020434219\ \overline{H}_i'')\ \overline{L}_i \overline{B}_i - 30{\cdot}503 \quad \text{.....(ix).}$$

We have then the following results:

Race	Predicted Value	Deviation from Observed	
Norwegians ...	1461	+ 5	Root Mean Square Residual =6·10
Lapps ...	1466	− 7	
Maori ...	1421	+ 3	
Australians ...	1284	+ 6	
Eskimos ...	1516	−10	
Egyptians E...	1442	+ 2	

For practical purposes it is adequate to use

$$\overline{C} = \frac{(\cdot 3004\,\overline{H}_i' + \cdot 2043\,\overline{H}_i'')\,\overline{L}_i\overline{B}_i}{1000} - 30{\cdot}50 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{ix}^{\mathrm{bis}}),$$

where $\overline{H}_i'$ and $\overline{H}_i''$ are the mean internal vertical heights from basion and opisthion respectively, $\overline{L}_i$ is the mean of half the sum of the two maximum internal lengths, and $\overline{B}_i$ the maximum internal breadth.

Formula ($\mathrm{ix}^{\mathrm{bis}}$) will predict *for these six races* the cranial capacity with a mean residual almost half that of Wagner's, and about two-thirds of that of the "best" formula available when using Wagner's expression for the product. Are we then to say that we have obtained a satisfactory *interracial* formula? Personally, I should say most decidedly not. You cannot form a satisfactory interracial formula on the basis of six races, any more than an intraracial formula on six skulls*, even if they are wisely distributed. There is only one brachycephalic race—the Lapps—included, and no proper representation of Asia, Europe, America or Africa. Another group of six quite different races would probably give an entirely different equation. This is a problem which I would urge Dr Wagner to tackle; he would at least increase our data for internal cranial comparisons.

Pearson and Hoadley *demonstrated* for the first time that the internal diametral product did give a higher correlation with cranial capacity than the external, and they measured the amount of the difference on their series of Egyptians. Dr Wagner has not worked out this difference for any of his five series, but he has contributed to our material for internal measurement. This is a valuable step in the right direction, but till the data are increased six- to tenfold on considerably longer series than some he has used, we ought not to set about finding, still less speak of, an interracial formula. Indeed, before even this is attempted, there ought to be some standardisation of instrument used for internal measurement, and further a standardisation of technique in measurement, especially that of the capacity. I am inclined to think, for example, that Dr Wagner is getting on his series lower capacities than other craniologists would do in using apparently the same method. What is more, I do not believe many craniologists would accept his process of deducing the internal "ear-bregma" height H_i by subtracting from the external "ear-bregma" height, a quantity he terms the "thickness of the parietal bone," which he does not measure at the bregma. Even if he did measure it at the bregma, it does not follow that the external "ear-bregma" line cuts the cranial vault at right angles. He does not make use of the quantity he terms the glabellar thickness of the frontal bone, but I am inclined to think that his process does not measure this thickness. I hold, however, that standardisation of measurements is essential before more work is done, and that most probably the $\overline{L}_i$, $\overline{B}_i$ and H_i', H_i'' (all measured by Dr Wagner, but the latter two not used) will be easier to accurately determine, and give better results than using B_f and H_i as finally adopted by him.

3. Lastly, I should like to say a few words on Dr Lee's and my external diametral product prediction formulae. They give as closely as it is possible to give a prediction formula with the $\overline{L}\,\overline{B}\,\overline{H}$ product *for the races on which they are based.* Thus (ii) give root mean square residuals $= 36{\cdot}16$ cm.^{3} and $35{\cdot}92$ cm.^{3}, and (iii) $22{\cdot}17$ cm.^{3} and $35{\cdot}92$ cm.^{3} respectively.

The same formulae applied to Dr Wagner's five races give:

(ii) $41{\cdot}02$ cm.^{3} and $33{\cdot}09$ cm.^{3}, as against $36{\cdot}16$ cm.^{3} and $35{\cdot}92$ cm.^{3};

(iii) $41{\cdot}70$ cm.^{3} and $33{\cdot}09$ cm.^{3}, as against $22{\cdot}17$ cm.^{3} and $35{\cdot}92$ cm.^{3}.

* The six skulls have an advantage over the six races. With caution their measurements may be accurately made, and the prediction formula follows by the usual statistical procedure. But with racial means a new factor enters into the problem, namely the size of the samples of each race, and this will affect the A and B constants of the prediction formulae, even if the individual crania of each race be measured with extreme accuracy.

While the female root mean square residuals have actually been decreased the male root mean square residuals, as we should anticipate, have been considerably increased, when we apply the formulae to the data for races on which they were not computed. But supposing our formulae were correctly applied, would a prediction with a root mean square residual of 41 cm.^{3} be valueless? Is it worth our while really striving at a root mean square residual of 9 or even 6 cm.^{3}? I doubt it, if two observers, owing to difference of technique, to varying size of sample, or to the effect of local races, may differ by 20 to 60 cm.^{3} in their direct measures of capacity!

Thus far I have assumed that Dr Wagner has applied our formulae correctly; but he has not done so. He has replaced our $H=OH$ of the *Frankfurter Verständigung* by what he terms the "ear-bregma" distance. He does not state how he measures this or how he found his poria. But assuming these to be determined by the skull on the craniophor, our OH is not his "ear-bregma" height, as he supposes. It may in some races be as much as 2—3 mm. greater (e.g. in the Maori or the Tibetan skulls), or there may be practically equality as in the case of the type English skull. It all depends on the slope of the cranial roof from apex to bregma. 1—3 millimetres may not appear much regarded in itself, but it modifies most sensibly P the product. For example, taking the Lapps, if the OH exceed by 2·5 mm. the ear-bregma distance in males, then formula (ii) would predict the value 1466 instead of 1419, and be in error -7 instead of -54, while if the female OH exceeded by 0·8 mm. the ear-bregma distance, the predicted capacity would have been 1301 instead of 1294, showing no error instead of -7. I lay no stress on these illustrations; I only intend to indicate that the actual OH cannot be measured by the "ear-bregma" height; the difference depends on how the apex-bregma stretch of the crown slopes and how near together the apex and bregma are situated. We cannot therefore apply legitimately formulae (ii) and (iii), using the "ear-bregma" distance. We can only test what an external diametral formula will achieve by working one out for Wagner's data. I have worked out the best male prediction formulae for the above six races, when we take as our products of external measurements

$$P_i''' = \bar{L}\bar{B}(\overline{OH'}) \text{ and } P_i^{iv} = \bar{L}\bar{B}H',$$

where OH' is not our vertical height OH, but the "ear-bregma" distance, and H' is not our vertical basion height H, but the basion bregma distance. All these external characters are provided by Wagner for his series.

We form the following table:

Race	$\bar{L}$	$\bar{B}$	OH'	P_i'''	H'	P_i^{iv}
Norwegians ...	187·70	142·34	113·28	302,6526·455	132·81	354,5313·723
Lapps ...	179·17	146·52	113·02	296,6999·729	129·05	338,7819·103
Maori ...	186·81	137·88	116·06	298,9399·527	136·63	351,9228·479
Australians ...	183·30	130·77	113·85	272,9000·553	135·92	325,8021·565
Eskimos ...	190·44	137·56	119·56	313,2104·5	139·39	365,1589·571
Egyptians E...	185·33	139·16	112·43	290,2757·630	133·51	344,7008·549

Now before going further we note:

(i) That the Lapp P_i''' and P_i^{iv} are less than those for the Norwegians. Accordingly, the predicted capacity will inevitably be less for the Lapps than for the Norwegians; the observed capacity for the former is greater than for the latter.

(ii) The Maori P_i''' and P_i^{iv} are greater than those for the Lapps. Accordingly, the predicted capacities for the Maori will be greater than for the Lapps; the observed capacity for the former is considerably less than for the latter.

(iii) The Egyptian E P_i''' and P_i^{iv} are less than those for the Maori, but the observed capacity of the latter is less than for the Egyptians.

These results indicate that the prediction formulae based on P_i''' and P_i^{iv} will not be very satisfactory in reproducing the capacities said to be observed. I obtained the following prediction formulae using:

OH', the "ear-bregma" length,

$$\bar{C}_i = \cdot 00057622174\, P_i''' - 272\cdot 514 \quad \ldots\ldots\ldots\

Race	Recorder	$\bar{C}$	$\bar{L}$	$\bar{B}$	H'	P_4	Predicted Capacity
Lapps	Wagner* (63)	1473	179·17	146·52	129·05	338,7819·103	1391
"	Schreiner† (151) ...	1465	178·3	147·0	128·1	335,7513·810	1376
"	Hällsten‡ (66)	1522	180·7	147·0	129·5	343,9895·550	1417
Australians (?)	Wagner* (13)	1278	183·30	130·77	135·92	325,8021·565	1325
" (Queensland)	Morant§ (19)	1288	185·1	131·1	133·3	323,4739·113	1314
" (General) ...	Morant§ (146)	1295	187·8	132·2	138·1	342,8630·796	1412
Maori	Wagner* (16)	1418	186·81	137·88	136·63	351,9228·479	1457
"	Scott‖ (43)	1476	185·5	140·1	137·6	357,6024·480	1486
Eskimos	Wagner* (18)	1526	190·44	137·56	139·39	365,1589·571	1524
" (Greenland)	Hrdlička¶ (34)	1559	190·5	135·9	139·5	361,1508·525	1504
"	Fürst and Hansen¶ (175)	1528	188·4	134·4	138·2	340,6486·272	1400

Considering the Australians, Morant's general series suggests by its diameters a series of crania larger by 83 cm.^{3} than Wagner's, and this is based not on 13 but 146 skulls, but the observed capacities only differ by 17 cm.^{3}. The Queensland and Wagner's series of about the same size suggest that the former falls short of the latter by 11 cm.^{3}, but the observed capacities differ in the inverse way by 10 cm.^{3}. This again suggests, it does not of course prove, that Wagner's value is too small.

Lastly, turning to the Lapp data, we note that Schreiner's and Wagner's, although differing much in predicted and observed values, give very close differences of observed and of predicted values, 8 cm.^{3} and 15 cm.^{3}. This is as it should be, if Wagner is a member of Schreiner's school. But if we turn to Hällsten's measurements, the latter's crania are larger for all diameters, and give a predicted value exceeding Wagner's by 26 cm.^{3}, yet the actually observed values are such that Hällsten's exceed Wagner's by 49 cm.^{3}.

Assuming, as I think we may, that the measured diameters will not differ largely by differences of personal equation in measurements, the above results do appear to indicate that Dr Wagner's technique in determining capacity gives results falling short of that of other observers; possibly, though I cannot prove it, they packed tighter. But what I do want to emphasise is this, that until we have standardisation in measurement of capacities, until we have such large series that errors of random sampling are negligible, and until we have such careful specification of the source of cranial series that we can define local races, various observers will give for the same "race" cranial capacities differing by any amount up to 60 cm.^{3}. In view of this, refined prediction formulae which provide a 6 cm.^{3} root mean square residual are scarcely to be preferred to rougher formulae with 20 to 30 cm.^{3} root mean square residuals. In other words, till the three conditions mentioned above are satisfied, I am still of the opinion that while internal measurements of the skull may lead us to interesting results as to the size characters of the brain, their value as leading to cranial capacity prediction is hardly worth the increased labour

* Memoir in this issue.

† *Zur Osteologie der Lappen*, Bd. II. S. 68. *Instituttet for Sammenlignende Kulturforskning*, Oslo, 1931.

‡ Hällsten: Matériaux pour servir à la connaissance des crânes des peuples Finnois. Bidrag till Kännedom af Finlands Natur och Folk *Finska Vetenskaps-Societeten*.

§ *Biometrika*, Vol. XIX. pp. 421, 429.

‖ *Biometrika*, Vol. XI. p. 98.

¶ *Annals of Eugenics*, Vol. I. pp. 261, 268.

of their determination. Overlooking for a moment the question of differences of technique, we may note that the standard error for capacity of a series of 13 crania is about 31 cm.^{3}; of a series of 16, 28 cm.^{3}; of a series of 18, 27 cm.^{3}; of a series of 63, 14 cm.^{3}; and of a series of 81, 13 cm.^{3}; and further that even deviations 50 °/$_0$ greater than these in excess or defect will occur in 13 to 14 °/$_0$ of samples of these sizes. What, I ask, is the value of using a formula with a 12 or even 6 cm.^{3} residual error on the *observed* small samples to get a predicted racial mean capacity when our sampling deviations alone are of the above order?

One further point I would like to emphasise strongly. Pearson and Hoadley found in the case of 729 male Egyptian crania the internal length of the right hemisphere greater than that of the left hemisphere. This was so opposed to the statements of some earlier investigators, notably Broca, who had asserted that the localisation of certain important centres found in the left hemisphere led to or must lead to a greater size of the left hemisphere, that Dr Woo investigated this point by direct measurements on the frontal and parietal bones of a number of races. His conclusions for 887 Egyptian male crania showed larger external measurements on the right frontal and parietal bones. This evidence Dr Wagner (see his p. 116) dismisses somewhat light-heartedly. We may add that Dr Woo has found in the following series (i) English (120), (ii) Australian (100), (iii) Southern Chinese (80), (iv) Punjabis (80), (v) Kanakas (60), (vi) Congo Negroes (50), and (vii) Nigerian Negroes (45), for the three measurements (*a*) ophryon to stepanion, (*b*) minimum arc bregma to asterion, (*c*) minimum arc sphenion to lambda with various degrees of difference preponderance in the size of the *right* frontal and parietal bones. It is needless to remark that these are the districts in which the active centres are stated to be localised on the *left* side and so to give it its predominance. This predominance of the left side Dr Wagner tells us "is confirmed by certain other investigations which Hoadley and Pearson do not mention in their survey of the literature." He then refers to papers by Hrdlička (1907) and Connolly (1932). The former might have been referred to had we known of it, but hardly the latter, as our paper was published three years before Connolly's! In neither of these papers is the procedure exactly comparable with the measurements taken by Hoadley and Pearson, Dr Wagner, or Dr Woo, and for numbers Connolly's series are not comparable with those of Hoadley and Pearson. The contradictions are so great that one is almost compelled either to believe that there is really no predominance of either hemisphere, or to accept the impossible, namely that some investigators, when the skulls are in *norma basalis* with the frontal, or it may be the occipital, on their right, have wrongly designated their measurements, mistaking right for left hemisphere, or *vice versa*!

Finally with regard to the measurements made by Dr Wagner on the cranial contours published in this Journal and discussed by him on pp. 113—116 above, it is only fair to Dr Woo to remark that a similar investigation on the same contours was made two years ago by Dr Woo himself and will be found already published by the *Academia Sinica**. Personally I am doubtful as to this use of such contours, and certain that contours diagraphically drawn should not be included, for they do not represent points lying in one and the same plane, and accordingly the distances measured are not necessarily homologous lengths on left and right hand sides of a median line.

* "Evidence of the Asymmetry of the Human Skull. Derived from Contour Measurements," Ts'ai Yüan P'ei Anniversary Volume (Suppl. Vol. 1. *Bulletin of the Institute of History and Philology, Academia Sinica*, Peiping, 1933). Dr Woo states that "where the differences are real and not chance ones, it is always the right side which is the greater," and again that "the main conclusions of the present enquiry are in perfect accord with those of my earlier study, both having shown that the normal human skull is not symmetrical since the right side is slightly larger than the left in several of its parts."

SOME PROBLEMS IN THE ANALYSIS OF REGRESSION AMONG k SAMPLES OF TWO VARIABLES

By B. L. WELCH, B.A.

I. Introduction.

In a series of papers J. Neyman and E. S. Pearson have developed a method based on a special application of the principle of likelihood, the object of which is to determine the most suitable test criteria to employ in testing certain statistical hypotheses.

E. S. Pearson and S. S. Wilks have extended the method to two variables. k samples, Σ_t, of two variables x and y being given, each sample was assumed to be drawn from a normal bivariate population π_t specified by parameters a_t, b_t, σ_{xt}, σ_{yt}, and ρ_t; i.e. the joint distribution of x and y for the population π_t was assumed to be of the form

$$p(x,y) = \frac{1}{2\pi\sigma_{xt}\sigma_{yt}\sqrt{1-\rho_t^2}}\, e^{-\frac{1}{2(1-\rho_t^2)}\left\{\frac{(x-a_t)^2}{\sigma_{xt}^2} - \frac{2\rho_t(x-a_t)(y-b_t)}{\sigma_{xt}\sigma_{yt}} + \frac{(y-b_t)^2}{\sigma_{yt}^2}\right\}}.$$

Certain hypotheses about the population parameters were then considered*.

In problems of regression rather than correlation this initial assumption may not however be appropriate, and it is proposed in the present note to consider the application of these methods of approach to the case in which we are primarily concerned with the analysis of variation of y for a given value of x. In the type of problem under review the observations fall naturally into a number, say k, of groups, and within the tth group the deviations of y from the regression straight line $y = \beta_t x + \alpha_t$ are supposed normally distributed with standard deviation σ_t. Thus for the tth group the probability distribution of y for a given x is assumed to be of the form

$$p(y|x) = \frac{1}{\sqrt{2\pi}\,\sigma_t}\, e^{-\frac{1}{2\sigma_t^2}(y-\beta_t x-\alpha_t)^2} \qquad \text{.....(1).}$$

In Fig. 1 a situation is shown where there are three groups, the sets of observations (x, y) for these being drawn for convenience so that they fall clear of one another.

The value of y that appears with any particular x in group t is regarded as a random deviate from a normal distribution whose mean is $\beta_t x + \alpha_t$ and standard deviation σ_t, where σ_t, β_t, and α_t are independent of x and are thus characters of the whole group t. Taking together all the values of x that can possibly occur in the group

* E. S. Pearson and S. S. Wilks: *Biometrika*, Vol. xxv. 1933, pp. 353—78.

a number of means $\beta_t x + \alpha_t$ are obtained which all lie on the line $Y = \beta_t X + \alpha_t$. These population regression lines are shown in Fig. 1 ($t = 1, 2, 3$). It should be noted that nothing has been said about the distribution of x, which need not even be continuous. Thus the array means do not necessarily lie continuously along the regression line. It follows that if, in a particular problem, the assumption of equation (1) can be made legitimately for certain x's in the group but not for others, then conclusions drawn from the criteria obtained below will still be valid within the restricted area defined by the permissible x's.

Our purpose is to apply the method of Neyman and Pearson to determine from the observed data criteria appropriate to test a variety of hypotheses regarding the group parameters σ_t, β_t, and α_t. One such hypothesis may be illustrated by con-

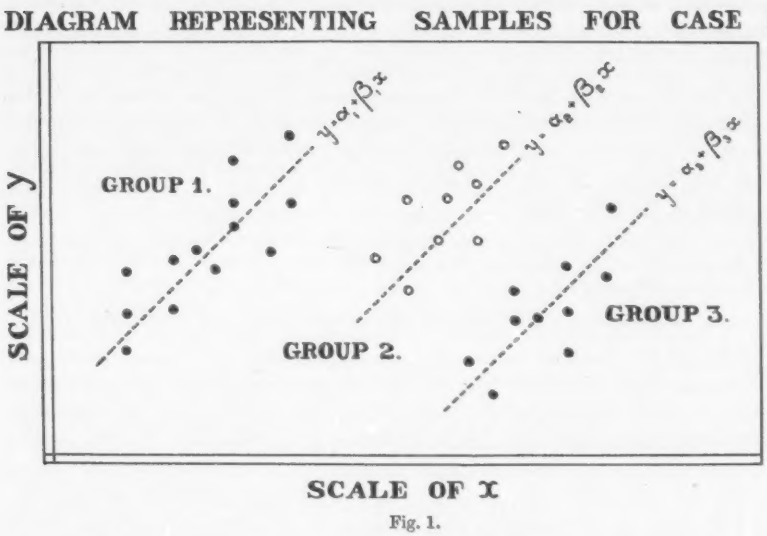

Fig. 1.

sidering the following example. An investigation into the heights of schoolchildren in Glasgow indicated that within certain limits of age, the regression of height (y) on age (x) both for boys and for girls could be regarded as linear, and that for a given age the distribution of height could in each case be taken as normal with standard deviation the same for each age considered. Clearly here the distribution of x need not concern us. All that we need to know is the range of age within which the assumption of linearity can be made. Assuming that such relations hold for a number of different groups of children (they may belong to different schools or social classes or races) we might ask whether the same regression line can be used to predict height from age for all groups, and also whether the variance about this regression line can be regarded as being the same throughout. This is in effect asking whether σ_t, β_t, and α_t are the same for all t.

Another situation which may be tested is shown in Fig. 1 where $\beta_1 = \beta_2 = \beta_3$, but α_t is not constant. The regression lines here are parallel but not coincident.

The method employed below leads for each hypothesis considered to a different test criterion λ. The sampling distribution to which this criterion is referred is that which it would follow (were the hypothesis tested true) if in repeated sampling the values of x were kept fixed, so that the only variation is that of y. It will be seen, however, that this sampling distribution of λ is independent of the x's. Thus, provided the hypothesis tested concerns only relations between σ_t, β_t, and α_t which do not contain x, the condition of fixed x's can be removed and the test of significance can be regarded as being made for any set of x's subject only to equation (1) being satisfied.

II. THE METHOD.

The observations in each group may be described as forming a sample $\Sigma_t (t = 1, 2, \ldots k)$ of (x, y). As stated above, this sample will be regarded rather as a sample of y, in which each y corresponds to a given x. It will then be found convenient to speak of a population π_t from which Σ_t has been drawn, meaning by this that for each x in Σ_t the corresponding y is distributed normally about $\beta_t x + \alpha_t$ with standard deviation σ_t. π_t is then specified by the parameters σ_t, β_t, and α_t.

The probability function for the joint occurrence of the k samples Σ_t is obtained by multiplying together expressions of the type given in equation (1). If (x_{ti}, y_{ti}) is the ith individual of the sample Σ_t, n_t the number of individuals in that sample and $N = \sum_1^k (n_t)$, then this function may be written

$$p = \left(\frac{1}{\sqrt{2\pi}}\right)^N \prod_1^k \left(\frac{1}{\sigma_t^{n_t}}\right) e^{-\frac{1}{2}\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where

$$\theta = \sum_{t=1}^{k} \sum_{i=1}^{n_t} \left\{\frac{(y_{ti} - \beta_t x_{ti} - \alpha_t)^2}{\sigma_t^2}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

It will be convenient to denote by A, B, C, D the following conditions:

A that included in equation (1), namely that

$$p(y \mid x) = \frac{1}{\sqrt{2\pi}\,\sigma_t} e^{-\frac{1}{2\sigma_t^2}(y - \beta_t x - \alpha_t)^2},$$

B that σ_t is the same for each group, i.e.

that $\sigma_t = \sigma \quad (t = 1, 2, \ldots k)$,

C that $\beta_t = \beta \quad (t = 1, 2, \ldots k)$,

D that $\alpha_t = \alpha \quad (t = 1, 2, \ldots k)$.

In addition it will be necessary to express algebraically the condition that the population means of the groups are collinear. If $\bar{x}_t$ and $\bar{y}_t$ are the means for the

n_t observations in the tth group and the symbol [ ] is used to denote expected or population value, then

$$[\bar{y}_t] = \beta_t \bar{x}_t + \alpha_t \qquad (t = 1, 2, \ldots k),$$

and the condition that the point $(\bar{x}_t, [\bar{y}_t])$ lie on the line

$$y = lx + m$$

is that $$\beta_t \bar{x}_t + \alpha_t = l\bar{x}_t + m \qquad (t = 1, 2, \ldots k).$$

This will be called condition E.

Neyman and Pearson's method affords a simple rule for obtaining appropriate test criteria once two sets of conditions have been defined. These are: (*a*) the conditions which can be assumed to be satisfied, and (*b*) the conditions which define the hypothesis to be tested. For instance, suppose that in addition to the assumption implied in equation (1) we can assume that σ_t is the same for each group and we wish to test whether in addition all the β_t's are equal. Then, using the notation given above, the set of conditions (*a*) consists of A and B taken together and the set (*b*) of A, B, and C. The conditions (*a*) define a class Ω of admissible sets of populations π_t, and the conditions (*b*) define a sub-class ω, of Ω, to which the set π_t must belong if the hypothesis tested be true.

The maximum value of p (equation (2)), when the population parameters vary so that the set π_t belongs to Ω, is called $p\,(\Omega \text{ max.})$. The maximum value when the set π_t is restricted to ω is called $p\,(\omega \text{ max.})$. The likelihood of the hypothesis has then been defined to be

$$\lambda = \frac{p\,(\omega \text{ max.})}{p\,(\Omega \text{ max.})}.$$

The greater the value of λ the more likely is it judged that the set π_t is restricted to ω, i.e. that the hypothesis tested is true. The scale of λ on which judgment is based is obtained by referring λ to its sampling distribution when the hypothesis is true.

Much of the value of this method of approach lies in the fact that an essential first step in obtaining λ is the definition of the classes Ω and ω, this requirement encouraging clear thinking in the process of analysis.

III. Types of Hypotheses Considered and Notation Used.

The hypotheses that will be tested, regarding the regression of y upon x, may be divided into three classes analogous to those which Neyman and Pearson denoted by H, H_1, and H_2*. Firstly there is the hypothesis H, that σ_t is the same for every group, i.e. that the variance of y about the regression lines is the same throughout the whole range of data. This is the hypothesis that condition B above is true. Then there are several hypotheses of the H_2 type concerning the values of α_t and β_t (conditions C, D or E), in testing which H_1 is accepted as true. For each

* J. Neyman and E. S. Pearson, "On the Problem of k Samples," *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A, 1931.

of these it is found that the present method leads to one of R. A. Fisher's tests in the analysis of variance. Finally there are hypotheses of the H type where nothing more is assumed than the condition A above and the hypothesis tested concerns both condition B and one or more of the conditions C, D, and E. The likelihood criteria for this type of hypothesis and for the H_1 hypothesis introduces into the analysis the geometric mean of sums of squares.

In developing the tests the following notation will be used. $\bar{x}_0$ and $\bar{y}_0$ are the means for the whole $N = \sum_t (n_t)$ observations, $\sum_i$ indicates summation for all observations (x_{ti}, y_{ti}) within the tth group, $\sum_t$ indicates summation for all groups, and $S = \sum_t \sum_i$.

$$\left.\begin{aligned} b_t &= \frac{\sum_i (x_{ti} - \bar{x}_t)(y_{ti} - \bar{y}_t)}{\sum_i (x_{ti} - \bar{x}_t)^2}, \qquad b_a = \frac{S(x_{ti} - \bar{x}_t)(y_{ti} - \bar{y}_t)}{S(x_{ti} - \bar{x}_t)^2}, \\ b_m &= \frac{S(\bar{x}_t - \bar{x}_0)(\bar{y}_t - \bar{y}_0)}{S(\bar{x}_t - \bar{x}_0)^2}, \qquad b_0 = \frac{S(x_{ti} - \bar{x}_0)(y_{ti} - \bar{y}_0)}{S(x_{ti} - \bar{x}_0)^2}. \end{aligned}\right\} \quad \ldots\ldots\ldots(4)$$

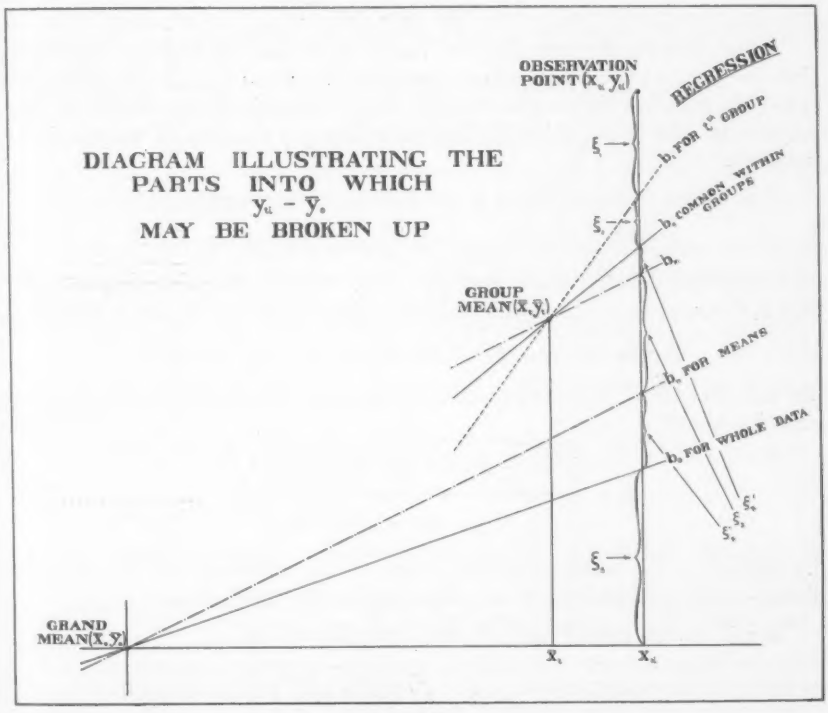

Fig. 2.

Thus b_t is the regression coefficient in the tth group, and b_a an average value obtained from within groups on the assumption that the k values of b_t do not differ significantly. b_m is the coefficient obtained from the weighted mean of the groups, and b_0 is obtained from pooling together the whole N observations. b_0 may also be regarded as a weighted average of b_a and b_m.

The corresponding regression lines are represented diagrammatically in Fig. 2, which shows also the parts into which $y_{ti} - \bar{y}_0$ may be broken, as follows:

$$\left.\begin{aligned} \xi_1 &= (y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t), & \xi_4' &= (b_a - b_m)(x_{ti} - \bar{x}_t), \\ \xi_2 &= (x_{ti} - \bar{x}_t)(b_t - b_a), & \xi_4'' &= (b_m - b_0)(x_{ti} - \bar{x}_0), \\ \xi_3 &= (\bar{y}_t - \bar{y}_0) - b_m(\bar{x}_t - \bar{x}_0), & \xi_5 &= b_0(x_{ti} - \bar{x}_0). \end{aligned}\right\} \quad \ldots\ldots\ldots(5)$$

The parts ξ_1, ξ_2, ξ_3, ξ_4 $(= \xi_4' + \xi_4'')$, and ξ_5 can be readily shown to be orthogonal, the product of any pair summed over the whole N observations vanishing. Hence, for example,

$$S(\xi_1 + \xi_2 + \xi_3 + \xi_4)^2 = S(\xi_1^2) + S(\xi_2^2) + S(\xi_3^2) + S(\xi_4^2) \quad \ldots\ldots\ldots(6).$$

IV. The Test for H.

As an example of a hypothesis of the H type, take the inclusive hypothesis that the populations π_t have $\sigma_t = \sigma$, $\beta_t = \beta$, and $\alpha_t = \alpha$ $(t = 1, 2, \ldots k)$, i.e. that there is no differentiation whatever between the form of regression in the different groups. Suppose we wish to test this hypothesis assuming only the form of the probability functions.

Ω is defined by the condition A and p (Ω max.) occurs when

$$\beta_t = b_t;\ \alpha_t = \bar{y}_t - b_t\bar{x}_t;\ n_t\sigma_t^2 = \sum_i \{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2.$$

The conditions defining ω are A, B, C, and D. Thus to find p (ω max.) we write $\beta_t = \beta$, $\alpha_t = \alpha$, and $\sigma_t = \sigma$ in (2) and maximise with respect to σ, β, and α. This gives

$$\beta = b_0;\ \alpha = \bar{y}_0 - b_0\bar{x}_0;\ N\sigma^2 = S\{(y_{ti} - \bar{y}_0) - b_0(x_{ti} - \bar{x}_0)\}^2.$$

On substituting we find $\lambda = p(\omega \text{ max.})/p(\Omega \text{ max.})$, or using the more convenient criterion $L = \lambda^{2/N}$,

$$L = \frac{\sqrt[N]{\prod_1^k \left[\frac{1}{n_t}\sum_i \{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2\right]^{n_t}}}{\frac{1}{N}S\{(y_{ti} - \bar{y}_0) - b_0(x_{ti} - \bar{x}_0)\}^2} \quad \ldots\ldots\ldots(7),$$

i.e. the ratio of the weighted geometric mean of the variance about the regression lines b_t to the total variance about a line fitted to the whole data.

It will be seen from (5) and (6) that L may be written

$$L = \prod_t \left(\frac{N}{n_t}\right)^{\frac{n_t}{N}} \frac{\sqrt[N]{\prod_1^k \left[\left(\sum_i \xi_1^2\right)^{n_t}\right]}}{S(\xi_1^2) + S(\xi_2 + \xi_3 + \xi_4)^2}.$$

If the hypothesis H is true, it is known* that

$S(\xi_1+\xi_2+\xi_3+\xi_4)^2/\sigma^2$ is distributed as χ^2 with $N-2$ degrees of freedom,

$\sum_i (\xi_1^2)/\sigma^2$ is distributed as χ^2 with n_t-2 degrees of freedom,

and $S(\xi_2+\xi_3+\xi_4)^2/\sigma^2$ is distributed as χ^2 with $2k-2$ degrees of freedom.

It follows that L may be written in the form

$$L = \prod_t \left(\frac{N}{n_t}\right)^{\frac{n_t}{N}} \frac{\sqrt[N]{\prod_t (\chi_t^2)^{n_t}}}{\sum_t (\chi_t^2)+\chi_q^2},$$

where the $k+1$ values of χ^2 are independently distributed†.

Following the lines of Neyman and Pearson's proof for the case of a single variable it can be shown that the pth moment coefficient of L about zero is

$$\mu_p' = \frac{N^p}{\prod_1^k \left\{n_t^{\frac{pn_t}{N}}\right\}} \frac{\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-2}{2}+p\right)} \prod_t \left\{\frac{\Gamma\left(\frac{n_t-2}{2}+\frac{pn_t}{N}\right)}{\Gamma\left(\frac{n_t-2}{2}\right)}\right\} \quad \ldots\ldots\ldots\ldots(8).$$

If the number of observations in each group is the same, i.e. $n_t = n$, then

$$\mu_p' = \frac{k^p\,\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-2}{2}+p\right)} \left\{\frac{\Gamma\left(\frac{n-2}{2}+\frac{p}{k}\right)}{\Gamma\left(\frac{n-2}{2}\right)}\right\}^k \quad \ldots\ldots\ldots\ldots(8').$$

As in the case of a single variable, it is probable that the sampling distribution of L can here again be represented approximately by the Type I distribution

$$p(L) = \frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\,\Gamma(m_2)} L^{m_1-1}(1-L)^{m_2-1} \quad \ldots\ldots\ldots\ldots(9),$$

the constants being calculated as follows by the method outlined by the above two writers‡:

$$m_1 = \frac{\mu_1'(\mu_1'-\mu_2')}{\mu_2'-\mu_1'^2}, \quad m_2 = \frac{(1-\mu_1')(\mu_1'-\mu_2')}{\mu_2'-\mu_1'^2} \quad \ldots\ldots\ldots\ldots(10),$$

where μ_1' and μ_2' are given by (8).

The probability that L is less than some specified value L_0 may then be found

* A general theorem from which it follows that these sums of squares are distributed independently in χ^2, is proved by R. A. Fisher in *Metron*, Vol. v. 1925. A proof following somewhat different lines is given by W. G. Cochran in *Proc. Camb. Phil. Soc.*, Vol. xxx. Part II. 1934.

† In the present case χ_t^2 and χ_q^2 have respectively n_t-2 ($t=1, 2, \ldots k$) and $2k-2$ degrees of freedom: in Neyman and Pearson's case the degrees of freedom were n_t-1 and $k-1$ respectively. Their proof will be found in pp. 468—471, *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A, 1931.

‡ *Loc. cit.* and also in *Biometrika*, Vol. xxiv, p. 415.

from the Incomplete B-function Tables*. If only the 5 °/₀ and 1 °/₀ points of L are wanted, they may be obtained by interpolating into R. A. Fisher's z-tables with

$$f_1 = 2m_2, \quad f_2 = 2m_1,$$

and then using the transformation $L = f_2/(f_2 + f_1 e^{2z})$, where the letters f_1 and f_2 are used to denote degrees of freedom in place of Fisher's n_1 and n_2.

V. The Test for H_1.

The class Ω is now defined by condition A, and ω by conditions A and B, i.e the test is whether σ_t is constant for all groups. p (Ω max.) occurs when

$$\beta_t = b_t; \ a_t = \bar{y}_t - b_t \bar{x}_t; \ n_t \sigma_t^2 = \sum_i \{(y_{ti} - \bar{y}_t) - b_t (x_{ti} - \bar{x}_t)\}^2.$$

p (ω max.) occurs when

$$\beta_t = b_t; \ a_t = \bar{y}_t - b_t \bar{x}_t; \ N\sigma^2 = S\{(y_{ti} - \bar{y}_t) - b_t (x_{ti} - \bar{x}_t)\}^2.$$

On substituting and writing $L_1 = \lambda_1^{2/N}$ we have

$$L_1 = \frac{\sqrt[N]{\prod_t \left[\frac{1}{n_t} \sum_i \{(y_{ti} - \bar{y}_t) - b_t (x_{ti} - \bar{x}_t)\}^2\right]^{n_t}}}{\frac{1}{N} S\{(y_{ti} - \bar{y}_t) - b_t (x_{ti} - \bar{x}_t)\}^2} \quad \ldots\ldots\ldots\ldots\ldots (11),$$

i.e. the ratio of the weighted geometric mean of the variance about the regression lines b_t to the weighted arithmetic mean of the same. Our only concern is now with the parts ξ_1 (Fig. 2), and when H_1 is true L_1 can be written in the form

$$L_1 = \prod_t \left(\frac{N}{n_t}\right)^{\frac{n_t}{N}} \frac{\sqrt[N]{\prod_t (\chi_t^2)^{n_t}}}{\sum_t (\chi_t^2)},$$

where each χ_t^2 is distributed in the known χ^2 form with $n_t - 2$ degrees of freedom ($t = 1, 2, \ldots k$).

The moments of L_1 can be found in the same way as those of Neyman and Pearson's single variate L_1, which is of the same form but with $n_t - 1$ degrees of freedom for χ_t^2. The pth moment of L_1 about zero is thus

$$\mu_p' = \frac{N^{Np}}{\prod_t \left\{n^{\frac{pn_t}{N}}\right\}} \frac{\Gamma\left(\frac{N-2k}{2}\right)}{\Gamma\left(\frac{N-2k}{2} + p\right)} \prod_t \left\{\frac{\Gamma\left(\frac{n_t - 2}{2} + \frac{pn_t}{N}\right)}{\Gamma\left(\frac{n_t - 2}{2}\right)}\right\} \quad \ldots\ldots\ldots (12),$$

and if the number of observations in each group is the same ($n_t = n$), then

$$\mu_p' = \frac{k^{Np}\, \Gamma\left(\frac{N-2k}{2}\right)}{\Gamma\left(\frac{N-2k}{2} + p\right)} \left\{\frac{\Gamma\left(\frac{n-2}{2} + \frac{p}{k}\right)}{\Gamma\left(\frac{n-2}{2}\right)}\right\}^k \quad \ldots\ldots\ldots\ldots\ldots (12').$$

* *Tables of the Incomplete Beta Function*, edited by Karl Pearson, *Biometrika* Office, University College, London.

As before, a Type I distribution is used as an approximation and m_1 and m_2 are calculated by substituting from (12) into (10). The Incomplete B-function Tables can then be used to give $p(L_1 < L_0)$.

In the case of a single variate tables have been given for certain points of the distribution of L_1 when the number of individuals in each sample is the same*. These tables may be used in the present test if they are entered with $n-1$ instead of n.

VI. THE TESTS FOR H_2.

The essential preliminary condition for testing hypotheses of the H_2 type is that σ_t should be the same for each group. If this is satisfied, θ of equation (3) can be written equal to ϕ/σ^2, where

$$\phi = S(y_{ti} - \beta_t x_{ti} - \alpha_t)^2.$$

Then
$$p = \frac{1}{(\sqrt{2\pi})^N \sigma^N} e^{-\frac{\phi}{2\sigma^2}}.$$

Let ϕ (Ω min.) be the value of ϕ obtained by minimising ϕ for variation of the parameters such that the set of populations π_t belongs to Ω, and let ϕ (ω min.) be similarly defined for ω. Then for p (Ω max.) we must have

$$\phi = \phi\,(\Omega \text{ min.}),$$

and also
$$\frac{dp}{d\sigma} = 0 = \left(\frac{1}{\sqrt{2\pi}}\right)^N e^{-\frac{\phi}{2\sigma^2}} \left[\frac{1}{\sigma^N}\frac{\phi}{\sigma^3} - \frac{N}{\sigma^{N+1}}\right],$$

i.e.
$$N\sigma^2 = \phi\,(\Omega \text{ min.}).$$

Thus
$$p\,(\Omega \text{ max.}) = \left(\frac{1}{\sqrt{2\pi}}\right)^N \left\{\frac{N}{\phi\,(\Omega \text{ min.})}\right\}^{\frac{N}{2}} e^{-\frac{N}{2}}.$$

Similarly
$$p\,(\omega \text{ max.}) = \left(\frac{1}{\sqrt{2\pi}}\right)^N \left\{\frac{N}{\phi\,(\omega \text{ min.})}\right\}^{\frac{N}{2}} e^{-\frac{N}{2}},$$

whence
$$L_2 = \lambda_2^{\frac{2}{N}} = \left[\frac{p\,(\omega \text{ max.})}{p\,(\Omega \text{ max.})}\right]^{\frac{2}{N}} = \frac{\phi\,(\Omega \text{ min.})}{\phi\,(\omega \text{ min.})},$$

or is the ratio of two minimum sums of squares.

To take a particular instance, suppose that σ_t is the same for each group and it is proposed to test whether β_t, the slope of the regression line, is the same within each group. The class Ω is defined by conditions A and B, and ω by conditions A, B, and C. For Ω, ϕ may be split up as follows:

$$\phi\,(\Omega) = S\{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2 + S\{(b_t - \beta_t)(x_{ti} - \bar{x}_t)\}^2 + S\{\bar{y}_t - \beta_t \bar{x}_t - \alpha_t\}^2,$$

* P. C. Mahalanobis, *Sankhyā, The Indian Journal of Statistics*, Vol. 1. pp. 114 and 122. Additional rather fuller tables have been prepared for publication in the Department of Applied Statistics, University College, London.

the sums of squares having $N-2k$, k, and k degrees of freedom respectively. Similarly

$$\phi(\omega) = S\{(y_{ti}-\bar{y}_t)-b_t(x_{ti}-\bar{x}_t)\}^2 + S\{(b_t-b_a)(x_{ti}-\bar{x}_t)\}^2 + S\{(b_a-\beta)(x_{ti}-\bar{x}_t)\}^2 + S\{\bar{y}_t-\beta\bar{x}_t-\alpha_t\}^2,$$

these sums of squares having $N-2k, k-1, 1$, and k degrees of freedom.

Minimising ϕ in each case with respect to the parameters at our disposal, is seen to be equivalent to making the terms containing those parameters vanish. Thus

$$L_2 = \frac{\phi(\Omega \text{ min.})}{\phi(\omega \text{ min.})} = \frac{S\{(y_{ti}-\bar{y}_t)-b_t(x_{ti}-\bar{x}_t)\}^2}{S\{(y_{ti}-\bar{y}_t)-b_t(x_{ti}-\bar{x}_t)\}^2 + S\{(b_t-b_a)(x_{ti}-\bar{x}_t)\}^2}.$$

When β_t is the same for each group, L_2 follows exactly the distribution of equation (9) with $m_1 = (N-2k)/2$ and $m_2 = (k-1)/2$, and the Incomplete B-function Tables can be used to obtain the exact significance level. If we are interested only in the 5 % and 1 % levels we can ask if $S(\xi_2)^2 = S\{(b_t-b_a)(x_{ti}-\bar{x}_t)\}^2$ is significantly large compared with $S(\xi_1^2) = S\{(y_{ti}-\bar{y}_t)-b_t(x_{ti}-\bar{x}_t)\}^2$*. These two sums of squares are independent and the L_2 test is thus equivalent to one of R. A. Fisher's tests in the analysis of variance. We may proceed in fact by calculating

$$z = \tfrac{1}{2}\log_e\left\{\frac{v_1/f_1}{v_2/f_2}\right\},$$

where $v_1 = S(\xi_2^2)$, $v_2 = S(\xi_1^2)$,

$f_1 = k-1$, and $f_2 = N-2k$.

The significance of z may then be tested by using R. A. Fisher's tables.

In the hypothesis tested, which may be denoted by $H_{2,1}$, we have been concerned only with the slope of the regression lines within the different groups, i.e. in testing whether β_t is constant. We may, however, be interested in testing a variety of other hypotheses regarding α_t and β_t which are all of the H_2 type, i.e. for all of which conditions A and B of p. 147 are assumed satisfied. In each case certain further conditions are assumed and others define the hypothesis to be tested. The resulting criteria of the likelihood method are precisely those which flow from the application of the analysis of covariance technique, but a brief indication in summarised form of the solution in the case of four further hypotheses may be of interest.

$H_{2,2}$. Finding after applying the test no reason to doubt $H_{2,1}$, we may assume that β_t is constant and test the further hypothesis that the regression lines for all groups are coincident. This is $H_{2,2}$.

$H_{2,3}$ and $H_{2,4}$. If $H_{2,2}$ is not satisfied we may still test the hypothesis that the population mean y's for each group lie on a straight line when plotted against the mean x's. Two cases arise:

(*a*) $H_{2,3}$; when it is not assumed that β_t is constant, i.e. when $H_{2,1}$ is not true.

(*b*) $H_{2,4}$; when it is assumed that $H_{2,1}$ is true.

* The meaning of the comparison is brought out by Fig. 2. We are asking if the parts ξ_2 are of significance having regard to the magnitude of the parts ξ_1.

$H_{2,5}$. The hypothesis that the regression lines for all groups are coincident may also be reached after testing first $H_{2,1}$ and then $H_{2,4}$, i.e. starting from the assumption that $H_{2,4}$ is true.

Reference should be made to Fig. 3, which shows diagrammatically the arrangement of the group regression lines when the various hypotheses are true. For $H_{2,2}$ and $H_{2,5}$ the regression lines should have been drawn to coincide but they have been side-stepped a little in order that the groups might be distinguished. The conditions determining the classes Ω and ω for each H_2 test are set out formally in the second and third columns of Table I.

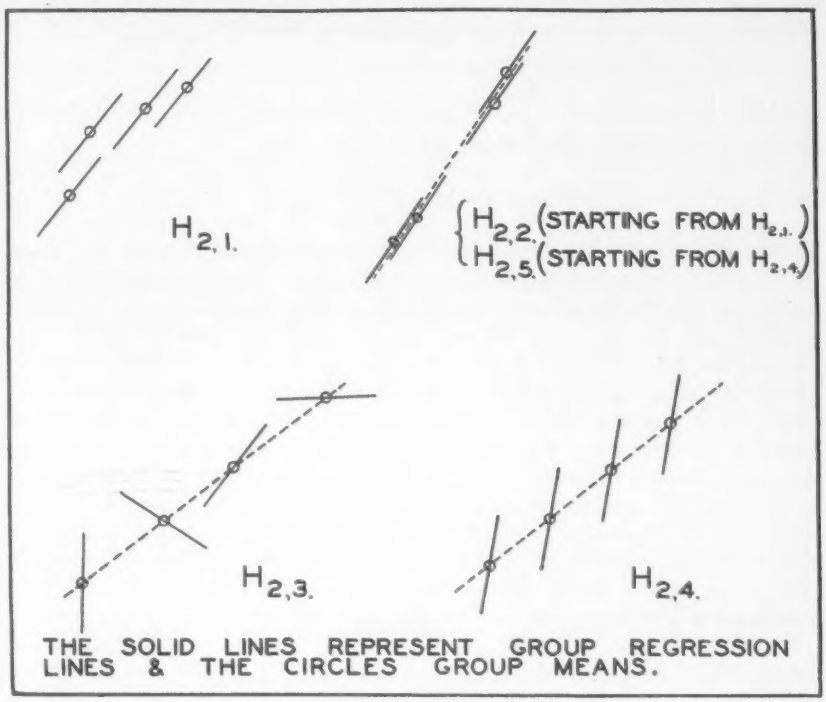

Fig. 3.

For each set of conditions the sum of squares $\phi = S\{y_{ti} - \beta_t x_{ti} - \alpha_t\}^2$ can be split up in a different way. For instance, when condition E holds, ϕ can in the first place be analysed into

$$\phi = S\{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2 + S\{(b_t - \beta_t)(x_{ti} - \bar{x}_t)\}^2 + S\{\bar{y}_t - \beta_t \bar{x}_t - \alpha_t\}^2.$$

The condition then permits the last sum of squares to be written

$$S\{\bar{y}_t - l\bar{x}_t - m\}^2,$$

and this breaks down further into

$$S\{\bar{y}_t - l\bar{x}_t - m\}^2 = S\{(\bar{y}_t - \bar{y}_0) - b_m(\bar{x}_t - \bar{x}_0)\}^2 + S\{(b_m - l)(\bar{x}_t - \bar{x}_0)\}^2 + S\{\bar{y}_0 - l\bar{x}_0 - m\}^2.$$

The algebra of the analysis of ϕ in other cases need not be set out here. The L_2 criteria for the different H_2 hypotheses have been shown to be given by

$$L_2 = \frac{\phi\,(\Omega \text{ min.})}{\phi\,(\omega \text{ min.})},$$

and it is seen that L_2 in each case can be written in the form $v_2/(v_1 + v_2)$, where v_1 and v_2 are independent sums of squares. The appropriate v_1 and v_2 and their respective degrees of freedom f_1 and f_2 are given in Table I*.

The exact significance level of L_2 can be found by using the Incomplete B-function Tables with $p = \frac{f_2}{2}$, and $q = \frac{f_1}{2}$ or alternatively, if we require only 5% and 1% levels of significance, we can use z, where

$$z = \tfrac{1}{2}\log_e\left\{\frac{v_1/f_1}{v_2/f_2}\right\}.$$

Something further should be added about the hypotheses $H_{2,3}$ and $H_{2,4}$ where the linearity of the means of the groups is considered. They differ from the others in that the relation being tested concerns not only the constants σ_t, β_t, and α_t but brings in also the mean x's. We are asking whether condition E can be true, i.e. whether $\beta_t\bar{x}_t + \alpha_t = l\bar{x}_t + m$ $(t = 1, 2, \ldots k)$. In the event of a positive answer being received to this question, it is not permissible to say that no matter what values $\bar{x}_t$ had been obtained the corresponding expected mean y's would be linear. Indeed if this were so, then it is obvious that we must have $\beta_t = l$ and $\alpha_t = m$, and thus have a common regression line for the whole data. It is however permissible to draw conclusions about the linearity of the means for any other sets of x's which lead to the same values $\bar{x}_t$, provided only that they satisfy the initial condition A. For we saw that the relation $\beta_t\bar{x}_t + \alpha_t = l\bar{x}_t + m$ enabled us to break down ϕ in such a way that it was possible to say that $v_1 = S\{(\bar{y}_t - \bar{y}_0) - b_m(\bar{x}_t - \bar{x}_0)\}^2$ was distributed as $\chi^2\sigma^2$ with $f_1 = k - 2$, and this takes no account of the way in which $\bar{x}_t$ is made up. It follows that the distribution of L_2, when condition E is satisfied, is independent of the partitioning of $\bar{x}_t$. Just as when testing relations which concern only σ_t, β_t, and α_t, it is possible to start by considering a fixed set of x's and then to remove this restriction, so in the present case owing to the nature of the distribution of the criterion we are able to extend our conclusions to any set of x's, subject only to the condition that $\bar{x}_t$ $(t = 1, 2, \ldots k)$ does not change, and that condition A remains satisfied.

* That v_1 and v_2 in each case are distributed as $\chi^2\sigma^2$ follows from the theorem already referred to. It will be found helpful to compare the sums of squares required for the different tests with the parts ξ_1, ξ_2 etc. shown in Fig. 2. A convenient form in which to calculate the sums of squares was set out by E. S. Pearson, in Appendix I to B. H. Wilsdon's paper, *Supplement to Journal of Royal Statistical Society*, Vol. I. p. 180. The notation there used is the same as that of this paper.

TABLE I. *Conditions defining Ω and ω and sums of squares entering into tests for hypotheses H_2.*

Hypothesis	Conditions defining Ω	Conditions defining ω	v_1	f_1	v_2	f_2
$H_{2,1}$	A, B	A, B, C	$S\{(b_t-b_a)(x_{ti}-\bar{x}_t)\}^2=S(\xi_2^2)$	$k-1$	$S\{(y_{ti}-\bar{y}_t)-b_t(x_{ti}-\bar{x}_t)\}^2=S(\xi_1^2)$	$N-2k$
$H_{2,2}$	A, B, C	A, B, C, D	$S\{(\bar{y}_t-\bar{y}_0)+b_a(x_{ti}-\bar{x}_t)-b_0(x_{ti}-\bar{x}_0)\}^2=S(\xi_3+\xi_4)^2$	$k-1$	$S\{(y_{ti}-\bar{y}_t)-b_a(x_{ti}-\bar{x}_t)\}^2=S(\xi_1+\xi_2)^2$	$N-k-1$
$H_{2,3}$	A, B	A, B, E	$S\{(\bar{y}_t-\bar{y}_0)-b_m(\bar{x}_t-\bar{x}_0)\}^2=S(\xi_3^2)$	$k-2$	$S\{(y_{ti}-\bar{y}_t)-b_t(x_{ti}-\bar{x}_t)\}^2=S(\xi_1^2)$	$N-2k$
$H_{2,4}$	A, B, C	A, B, E	$S\{(\bar{y}_t-\bar{y}_0)-b_m(\bar{x}_t-\bar{x}_0)\}^2=S(\xi_3^2)$	$k-2$	$S\{(y_{ti}-\bar{y}_t)-b_a(x_{ti}-\bar{x}_t)\}^2=S(\xi_1+\xi_2)^2$	$N-k-1$
$H_{2,5}$	A, B, C, E	A, B, C, D	$(b_a-b_m)^2\dfrac{\{S(\bar{x}_t-\bar{x}_0)^2\}\{S(x_{ti}-\bar{x}_t)^2\}}{S(x_{ti}-\bar{x}_0)^2}=S(\xi_4^2)$	1	$S\{(y_{ti}-\bar{y}_t)-b_a(x_{ti}-\bar{x}_t)\}^2$ $+S\{(\bar{y}_t-\bar{y}_0)-b_m(\bar{x}_t-\bar{x}_0)\}^2=S(\xi_1+\xi_2+\xi_3)^2$	$N-3$

TABLE II. *Summary of tests of crushing strength performed on normal concrete (y) and dry consistence mortar (x).*

Unit: lbs. per sq. in. Sums of squares and products given to nearest 1000.

Source	(1) Sum of Squares (y)	(2) Sum of Products (x and y)	(3) Sum of Squares (x)	(4) Regression Coefficient Col. (2) ÷ Col. (3)	(5) Col. (1) − Col. (4) × Col. (2)*
1 day ($t=1$)	3 323	7 368	16 781	$b_1=$ ·43907	88
3 days ($t=2$)	9 541	13 972	21 942	$b_2=$ ·63677	644
7 days ($t=3$)	7 920	11 326	20 327	$b_3=$ ·55719	1 609
28 days ($t=4$)	5 926	8 598	17 798	$b_4=$ ·48307	1 772
Within Ages	26 710	41 264	76 848	$b_a=$ ·536951	4 554
Between Ages	104 831	113 087	123 517	$b_m=$ ·915560	1 293
Whole Series	131 541	154 351	200 365	$b_0=$ ·770349	12 637

* Column (5) contains the sums of squares of the deviations from the respective regression lines, calculated most conveniently by noting that

$$\sum_i\{(y_{ti}-\bar{y}_t)-b_t(x_{ti}-\bar{x}_t)\}^2=\Sigma(y_{ti}-\bar{y}_t)^2-b_t\sum_i(y_{ti}-\bar{y}_t)(x_{ti}-\bar{x}_t),$$

and similarly for b_a, b_m and b_0.

If the question at issue is not whether the means $\bar{y}_t$ are linear but simply whether they differ significantly, then instead of condition E we shall have to test whether $\beta_t \bar{x}_t + \alpha_t = m\,(t = 1, 2, \ldots k)$. The sum of squares $S(\bar{y}_t - \beta_t \bar{x}_t - \alpha_t)^2$ will become

$$S(\bar{y}_t - m)^2 = S(\bar{y}_t - \bar{y}_0)^2 + S(\bar{y}_0 - m)^2$$

and the L_2 criteria will be modified accordingly, $S(\bar{y}_t - \bar{y}_0)^2$ with $f_1 = k - 1$ being used instead of $S(\bar{y}_t - \bar{y}_0) - b_m(\bar{x}_t - \bar{x}_0)\}^2$ with $f_1 = k - 2$. The restriction on the x's is the same as before.

VII. The Hypotheses H.

Only one test of the H variety was treated in § IV. Without going into much detail, it is of interest to note that other hypotheses of this type can be considered in much the same fashion as were the hypotheses H_2. The criterion L each time takes the same form:

$$L = \prod_t \left(\frac{N}{n_t}\right)^{\frac{n_t}{N}} \frac{\sqrt[N]{\prod_t (\chi_t^2)^{n_t}}}{\sum_t (\chi_t^2) + \chi_q^2} \qquad \text{.....(13).}$$

χ_q^2 is the only part that changes, and as it does the moments of equation (8) will be modified. In fact, if f is the number of degrees of freedom of χ_q^2, then the moments of L about zero, where there are n observations in each group, will be

$$\mu_p' = \frac{k^p\, \Gamma\left(\frac{N - 2k + f}{2}\right)}{\Gamma\left(\frac{N - 2k + f}{2} + p\right)} \left\{ \frac{\Gamma\left(\frac{n-2}{2} + \frac{p}{k}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \right\}^k \qquad \text{.....(14).}$$

The complete hypothesis considered above was a particular case where f was $2k - 2$. As one further instance of an H test, suppose that no other assumption than condition A is made and we wish to test whether conditions B and C hold, i.e. whether $\sigma_t = \sigma$ and $\beta_t = \beta$. This differs from the $H_{2,1}$ test in that constancy of σ_t is not now assumed, but is one of the elements to be tested. The L criterion is seen to be

$$L = \frac{\sqrt[N]{\prod_1^k \left[\frac{1}{n_t} \sum_i \{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2\right]^{n_t}}}{\frac{1}{N} S\{(y_{ti} - \bar{y}_t) - b_a(x_{ti} - \bar{x}_t)\}^2}.$$

Putting L into the form (13), we have

$$\chi_q^2 = \frac{S\{(b_t - b_a)(x_{ti} - \bar{x}_t)\}^2}{\sigma^2},$$

which has $k - 1$ degrees of freedom. The moments of L follow by putting $f = k - 1$ in (14), and then the 5 % and 1 % levels may be obtained by the method outlined on p. 152.

VIII. ILLUSTRATIVE EXAMPLE.

The data which will be used to illustrate the tests described above was collected at the Building Research Station. From the point of view of the H_2 tests, it has already been discussed by Mr B. H. Wilsdon. The problem concerns the relationship between crushing strength of concrete and mortar, and the particular data considered here are those on which his Table I (*c*)* is based. Corresponding tests were performed on dry mortar (x) and normal concrete (y) made from eight different cements ($n=8$, $i=1, \ldots 8$), and the results fall into four groups ($k=4$, $t=1, \ldots 4$) corresponding to four different ages of mortar and concrete. The relevant information is summarised in Table II.

First it may be asked whether the regression of concrete strength upon mortar strength can be represented by the same regression line for all four ages, and also at the same time whether the variance about that regression line is the same for each age. This is testing the complete hypothesis H. To calculate L, insert into (7) the values from Col. (5) of Table II, i.e. for $\sum_i \{(y_{ti}-\bar{y}_t)-b_t(x_{ti}-\bar{x}_t)\}^2$ ($t=1, 2, 3, 4$), 88, 644, 1609 and 1772 respectively, and for $S\{(y_{ti}-\bar{y}_0)-b_0(x_{ti}-\bar{x}_0)\}^2$, 12637. It is found that $L=\cdot2008$.

From (8) the moments of L are

$$\text{Mean} = \cdot7038874, \qquad \mu_2 = \cdot0128115, \qquad \text{i.e. } \sigma = \cdot1132.$$

Following the method given for approximating to the 5% and 1% levels of L, we have from (10)

$$m_1 = 9{\cdot}967092, \quad m_2 = 4{\cdot}19297, \quad f_2 = 19{\cdot}9342, \quad f_1 = 8{\cdot}3859,$$

whence interpolating in R. A. Fisher's z-tables,

$$\begin{cases} z_{\cdot05} = \cdot4435, & z_{\cdot01} = \cdot6302, \\ L_{\cdot05} = \cdot4947, & L_{\cdot01} = \cdot4026. \end{cases}$$

L is clearly significantly small and the composite hypothesis must be rejected.

We may now investigate more closely where the heterogeneity lies. To test only whether the variances are equal, i.e. the hypothesis H_1, the appropriate criterion is that of equation (11). Substituting the values from Table II, we have $L_1=\cdot6168$.

The lower 5% and 1% levels of L_1 may be obtained from the tables already referred to, or otherwise directly from (12) we have

$$\text{Mean} = \cdot8798590, \quad \mu_2 = \cdot0078003, \quad \text{i.e. } \sigma = \cdot08832,$$

giving $\qquad m_1 = 11{\cdot}0437, \quad m_2 = 1{\cdot}5080.$

* *Loc. cit.* p. 184. The figures given in the present paper are not exactly the same as those in Mr Wilsdon's table owing to the fact that the original figures as they were given to me were later altered. The discrepancies do not materially affect the conclusions reached.

Instead of proceeding by calculating f_1 and f_2 and interpolating into the z-tables, the exact significance level of L_1 may be found by evaluating

$$B_{\cdot 6168}(11{\cdot}0437, 1{\cdot}5080).$$

We obtain the chance that $L_1 < {\cdot}6168$ to be ${\cdot}0126$. It is thus unlikely that σ_t is constant for all ages.

Since the variances may differ there is doubtful justification in proceeding to test the hypotheses H_2, but the results are given in Table III for the sake of illustration*. z in each case is $\frac{1}{2}\log_e\left\{\frac{v_1/f_2}{v_2/f_2}\right\}$, where v_1 and v_2 are given in Table I and may be calculated from the figures of Table II.

TABLE III. *Summary of H_2 tests.*

Hypothesis	z	f_1	f_2	$z_{\cdot 05}$	$z_{\cdot 01}$	Significance of z
$H_{2,1}$	Negative	3	24	·5508	·7757	Not significant
$H_{2,2}$	1·3856	3	27	·5427	·7631	Significant
$H_{2,4}$	·6718	2	27	·6051	·8513	Doubtful
$H_{2,5}$	1·7585	1	29	·7155	1·0139	Significant

These tests would allow us to accept the equality of the regression coefficients for each group and perhaps the linearity of the means. The equality of the within groups regression coefficient (β) and the regression coefficient of the means (l), i.e. the hypothesis of a common regression line for the whole data, would have to be rejected.

In conclusion I should like to express my thanks to Dr E. S. Pearson for suggesting this problem to me and for the help he has given me at every stage of the work.

* The reader may be referred to a sampling experiment carried out by E. S. Pearson in the case of a single variable, which suggested that the H_2 test was not affected even by quite large changes in σ_t from group to group. See *Biometrika*, Vol. XXI. p. 346, 1929.

ON AN IMPORTANT CLASS OF STATISTICAL HYPOTHESES.

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1. Introductory.

Dr J. Neyman suggested to me as a topic for my thesis for a doctor's degree a problem which may be stated as follows.

Assume that it is known that each of the n independent variates

$$x_1, x_2, \dots x_n \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

follows the Gaussian law of frequency,

$$p(x_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i - m_i)^2}{2\sigma^2}} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2),$$

with a common standard deviation, σ. The mean values, m_i, of the variates (1) are connected with s unknown parameters

$$a_1, a_2, \dots a_r, a_{r+1}, \dots a_s \qquad (s < n) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(3)$$

by means of the formulae

$$m_i = c_{1i}a_1 + c_{2i}a_2 + \dots + c_{si}a_s \qquad (i = 1, 2, \dots n) \quad \dots(4),$$

where the coefficients c_{jk} are known and form a non-singular matrix

$$(c_{jk})_{\substack{j=1,2,\dots s \\ k=1,2,\dots n}} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(5).$$

The problem to be solved consists in testing the statistical hypothesis, H, that r known linear and independent functions of the parameters (3) have some definite given values, for instance that

$$b_{1i}a_1 + b_{2i}a_2 + \dots + b_{si}a_s = B_i^0 \qquad \text{for } i = 1, 2, \dots r \leqslant s \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(6),$$

where the coefficients b and the constant terms B^0 are known numbers.

It will be convenient to call a statistical hypothesis of the above type a linear hypothesis.

Different aspects of the problem considered have been already treated by many authors, as by A. A. Markoff (1), "Student" (2), R. A. Fisher (3, 4), J. Neyman and E. S. Pearson (5, 6) and by J. Neyman (7). The points of view of these authors were different. I shall treat the problem in its general form following the lines indicated by Neyman and Pearson (8, 9 and 10). It will be seen that some of the calculations I shall have to perform and some of the results I shall obtain are identical with those already published by the authors quoted above.

2. Examples.

Before I proceed to the solution of the problem it will perhaps be useful to illustrate its generality.

Example 1. Suppose that $s=1$, and that the equations (4) reduce to the following:

$$m_i = a_1 \qquad (i = 1, 2, \dots n) \quad \dots\dots\dots\dots\dots\dots (7).$$

In this case the hypothesis, H, reduces itself to that of "Student," namely that

$$a_1 = a_1^0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (8).$$

In fact the variates (1) may be considered as forming a sample for one unknown univariate normal population, π, and the hypothesis to test is that the mean of the population π is equal to a_1^0.

Example 2. Suppose that $s = 2$, while the equations (4) are

$$m_i = a_1 \qquad \text{for } i = 1, 2, \dots r \quad \dots\dots\dots\dots\dots\dots (9),$$

and

$$m_j = a_2 \qquad \text{for } j = r+1, r+2, \dots n \quad \dots\dots\dots\dots\dots\dots (10).$$

The hypothesis, H, that

$$a_2 - a_1 = B_1^0 = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (11)$$

(corresponding to the equations (6)) may now be formulated as follows. The variates (1) may be considered as forming two samples,

$$x_1, x_2, \dots x_r \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (12)$$

and

$$x_{r+1}, x_{r+2}, \dots x_n \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (13),$$

from two normal univariate populations π_1 and π_2, with common unknown variance σ^2, and the hypothesis tested consists in the assumption that the means of the populations π_1 and π_2 are equal. This is what we may call the generalised hypothesis of "Student."

Example 3. Assume $s > 2$,

$$n = n_1 + n_2 + \dots + n_s \qquad (n_i > 1) \quad \dots\dots\dots\dots\dots\dots (14),$$

and suppose that the equations (4) are

$$m_i = a_1 \qquad \text{for } i = 1, 2, \dots n_1,$$

and

$$m_j = a_2 \qquad \text{for } j = n_1 + 1, n_1 + 2, \dots n_1 + n_2,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (15),$$

generally

$$m_i = a_k \qquad \text{for } \sum_{j=1}^{k-1} n_j < i \leqslant \sum_{j=1}^{k} n_j,$$

where $k = 1, 2, \dots s$.

Consider the hypothesis, H, which may be expressed by means of $s-1$ equations of the form (6):

$$a_k - a_s = B_k^0 = 0 \qquad \text{for } k = 1, 2, \dots s-1 \quad \dots\dots\dots\dots\dots\dots (16).$$

It will be easily seen that the variates (1) may now be considered as forming s samples from s normal univariate populations $\pi_1, \pi_2, \dots \pi_s$, with a common standard deviation σ. The hypothesis to test is, that the means of all the populations are equal. This is the case considered by Neyman and Pearson (hypothesis H_2 in their notation) and also by R. A. Fisher in his analysis of variance. However it will be seen that the statement of the problem and the point of view of the solution will be different from those of Fisher.

Example 4. Put $s = 2$ and assume the equations (4) to be of the form

$$m_i = a_1 + a_2 y_i \quad (i = 1, 2, \dots n) \dots\dots\dots\dots (17),$$

where the numbers y_i are known.

We may now consider three important hypotheses:

H_1, assuming that $a_1 = a_1^0$ .....(18),

H_2, assuming that $a_2 = a_2^0$ .....(19),

H_3, assuming that $a_1 + a_2 Y = X$ .....(20),

where a_1^0, a_2^0, X and Y are given numbers.

The hypotheses H_1, H_2 and H_3 are connected with certain problems of correlation theory. Consider a population π of individuals with characters x and y and suppose that for each value of y the character x is normally distributed about its mean, say,

$$m_y = a_1 + a_2 y \dots\dots\dots\dots (21),$$

with the standard deviation σ, independent of the value of y. The population π may be considered as a set of populations, say π_y, each of them being determined by the value of the character y, common to all its elements. The variates (1) may now be considered as a sample drawn from the population π. If besides the values of (1) we know also the corresponding values of the character y,

$$y_1, y_2, \dots y_n \dots\dots\dots\dots (22),$$

where some of the y's may be equal, but at least two of them should be different, then we may consider each value of x as belonging to an individual, drawn at random from the partial population π_y.

The hypothesis H_1 assumes that the constant term in the regression equation (17) has a certain definite value. This hypothesis, in its application to some problems of agricultural experimentation, has been considered by J. Neyman (11).

The hypothesis H_2 concerns the value of the regression coefficient of x on y. Its test is to be found in the book by R. A. Fisher (12).

Finally the hypothesis H_3 has been dealt with by H. Schultz (13) and—in a more general way—by Miss K. Iwaszkiewicz and concerns the error of the ordinate of a fitted regression line. The result obtained by K. Iwaszkiewicz is published—with a suitable reference—in a paper by J. Neyman (7) showing its importance to some agricultural problems*.

The hypotheses considered in Example 4 may be easily generalised for the cases when the regression of x on y is not linear. This hypothesis in its general form is treated in Fisher's book (12).

Example 5. Put $s = 2r$ and assume

$$n = n_1 + n_2 + \dots + n_r \quad (r > 1) \dots\dots\dots\dots (23).$$

The equations (4) will be written

$$m_i = a_{2j-1} + a_{2j} y_i \quad \text{for} \sum_{k=1}^{j-1} n_k < i \leqslant \sum_{k=1}^{j} n_k \quad (j = 1, 2, \dots r) \dots\dots (24).$$

* The method of testing the hypothesis H_3 is given in § 8.

In applications we have sometimes to consider the hypothesis H_1 expressed by means of $r-1$ equations

$$a_{2k} - a_{2r} = B_k^0 = 0 \qquad (k = 1, 2, \dots r-1) \quad \dots(25),$$

and also the hypothesis H_2, that

$$\left.\begin{aligned} a_{2k-1} - a_{2r-1} = B^0_{2k-1} = 0 \\ a_{2k} - a_{2r} = B^0_{2k} = 0 \end{aligned}\right\} \qquad (k = 1, 2, \dots r-1) \quad \dots(26).$$

These hypotheses may be expressed in statistical terms as follows. Consider r populations $\pi_1, \pi_2, \dots \pi_r$ of individuals having some characters x and y. Suppose that the regression of x on y in each population π_i is known to be linear. Suppose further that, whatever the population π_i and whatever the specified value of y, the variance of x in the corresponding y-array is known to have the same (though unknown) value σ^2. The hypothesis H_1 consists in the assumption that the coefficients of regression of x on y in all r populations have equal values. The hypothesis H_2 is more stringent and assumes that not only the coefficients of regression but also the constant terms in the regression equations have a common value in all populations $\pi_1, \pi_2, \dots \pi_r$, thus that the regression lines of x on y are in all populations identical.

As far as I know these hypotheses were first considered by Miss K. Iwaszkiewicz (14), who showed their importance in agricultural experimentation and in some industrial research problems.

Further examples could be easily given, but I think that those which have been described above, together with those given in §§ 5 and 8, are sufficient to show the generality of the problem.

It is here assumed that, before applying the results of this paper to any group of experimental data, a test has been applied to show that the assumption of the standard deviation of the x's being constant is actually satisfied*.

3. Simplification of the Problem.

Before I proceed to the solution of the problem, I shall reduce it to a simpler form.

It may be shown that the problem of testing the hypothesis H, expressed by r independent linear equations with regard to the parameters a with known coefficients, can be reduced to that of testing the hypothesis, say H_1, ascribing to r of the parameters a some definite values. In fact, the equations (6) being independent, we may solve them with regard to some r parameters a. As the order in which the parameters are numbered is of no importance, we may assume that the solution gives

$$a_k = a_k(B_1^0, B_2^0, \dots B_r^0, a_{r+1}, a_{r+2}, \dots a_s) \qquad (k = 1, 2, \dots r) \quad \dots(27).$$

* This must either have been previously ascertained or else it must be done on the data themselves. In the case of bivariate populations this certainly would require that there should be enough observations of the x's with the same y to determine with reasonable accuracy the array variance.

Substituting the above values of a_k $(k = 1, 2, \ldots r)$ into the equations (4), we get similar linear equations, say,

$$m_i = c_{1i}' B_1^0 + c_{2i}' B_2^0 + \ldots + c_{si}' a_s \quad \ldots\ldots\ldots\ldots\ldots\ldots (28),$$

with known coefficients c'. Now it may be assumed that the means m_i are linear functions of the form (28) of s parameters

$$B_1, B_2, \ldots B_r, a_{r+1}, a_{r+2}, \ldots a_s \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

and that the hypothesis H determines the values of r of these parameters, namely,

$$B_k = B_k^0 \qquad (k = 1, 2, \ldots r).$$

Thus if we are able to test any hypothesis H_1, we shall be able to test also the hypothesis of the type H.

4. Solution based on the Principle of Likelihood.

I shall now give a solution of the problem of testing the hypothesis H_1, following the lines indicated by J. Neyman and E. S. Pearson (8), that is to say the solution based on the principle of likelihood (in the sense of the word used by Neyman and Pearson—not by R. A. Fisher). This is as follows. The first stage consists in stating Ω, or the set of admissible simple hypotheses.

We shall include in the set Ω any hypothesis, ascribing to the unknown parameters $\sigma, a_1, a_2, \ldots a_s$ any definite values, with the only restriction that $\sigma > 0$.

The second stage consists in defining the set ω of simple hypotheses belonging to the composite hypothesis, H_1, to be tested.

The set ω will contain any hypothesis which may be described by

$$a_1^0, a_2^0, \ldots a_r^0, a_{r+1}, a_{r+2}, \ldots a_s, \sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots (30),$$

where the a_i^0's are specified by the hypothesis tested, H_1, while $a_{r+1}, a_{r+2}, \ldots a_s$ and $\sigma > 0$ are arbitrary.

Further proceedings are as follows. We suppose that the observations provide us with values of the variates $x_1, x_2, \ldots x_n$. We have to calculate the simultaneous frequency function of these variates, determined by a simple hypothesis contained in Ω. Let it be $p(x_1, x_2, \ldots x_n)$. Obviously $p(x_1, x_2, \ldots x_n)$ will depend upon the values of parameters a_i and σ. We keep the x's constant, and choose out of the set Ω the simple hypothesis for which $p(x_1, x_2, \ldots x_n)$ is maximum. This absolute maximum of $p(x_1, x_2, \ldots x_n)$ will be denoted by p_Ω. Next we keep again x's constant, and choose out of the hypotheses included in ω the one for which $p(x_1, x_2, \ldots x_n)$ is maximum. This will be a relative maximum and will be denoted by p_ω.

According to the principle mentioned, the test of the hypothesis H_1 should be based on the values of the likelihood (in the sense of Neyman and Pearson), namely,

$$\lambda = \frac{p_\omega}{p_\Omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31).$$

If for any set of observational data λ is small, this means that the set of admissible hypotheses contains some of them, from the point of view of which the

facts observed are much more probable than from the point of view of the hypothesis tested. In these circumstances we are intuitively inclined to reject the tested hypothesis. In order to judge whether the value of λ is "small" or "large," we consider the probability P of rejecting unjustly the hypothesis tested, that is to say the probability of rejecting H_1, when it is true. In most practical cases we may agree upon the value of P which may be considered as "small." For instance, we may accept the view that it will do no great harm if we reject the hypothesis unjustly not oftener in the long run than say once in a hundred times. This arbitrarily chosen value of the probability of an error in rejecting the hypothesis H_1 has been termed by Neyman the confidence coefficient ϵ. The choice of ϵ determines the limit between "small" and "large" values of λ. Denote by

$$P\{(\lambda \leqslant \lambda_0) \mid H_1\}$$

the probability, determined by the hypothesis H_1, that the observations will give us the value of $\lambda \leqslant \lambda_0$, where λ_0 is any number between zero and unity. Choose now λ_0, so that we have

$$P\{(\lambda \leqslant \lambda_0) \mid H_1\} = \epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32),$$

and consider it as a limit between the values of λ "small" and "large." Obviously, if we agree to reject the hypothesis tested when $\lambda \leqslant \lambda_0$, the probability of rejecting H_1 unjustly will be

$$P = \phi(H_1)\, P\{(\lambda \leqslant \lambda_0) \mid H_1\} \leqslant P\{(\lambda \leqslant \lambda_0) \mid H_1\} = \epsilon \quad \ldots\ldots\ldots (33),$$

where $\phi(H_1) \leqslant 1$ denotes the unknown probability *a priori* of the hypothesis H_1.

Thus the test of the hypothesis H_1 requires the knowledge of the probability

$$P\{(\lambda \leqslant \lambda_0) \mid H_1\},$$

or of the frequency distribution of λ, determined by the hypothesis tested, H_1.

Having thus described the problem, we shall proceed to its solution.

The simultaneous frequency function of the variates (1) will be

$$p(x_1, x_2, \ldots x_n) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{\sum_{i=1}^{n}(x_i - m_i)^2}{2\sigma^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (34).$$

The process of maximising (34) with regard to the a_i and σ is simpler, if we try to maximise $\log p(x_1, x_2, \ldots x_n)$ instead of (34). Taking into account the equations (4) connecting the m's with the parameters a, we may write

$$\frac{\partial \log p}{\partial \sigma} = -\frac{n}{\sigma} + \frac{\chi^2}{\sigma^3} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35),$$

$$\frac{\partial \log p}{\partial a_j} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - c_{1i}a_1 - c_{2i}a_2 - \ldots - c_{si}a_s)\, c_{ji} = 0 \quad \ldots\ldots\ldots (36),$$

where $$\chi^2 = \sum_{i=1}^{n} (x_i - m_i)^2 = \sum_{i=1}^{n} (x_i - c_{1i}a_1 - c_{2i}a_2 - \ldots - c_{si}a_s)^2 \quad \ldots\ldots\ldots (37).$$

Obviously the equations (36) are those which we should get if we wanted to minimise χ^2. The possibility of their solution is guaranteed by the condition concerning the matrix (5).

Denote by nS_a^2 the minimum value of χ^2 (37). This will be obtained by solving (36) with regard to the a's. Denote by

$$a_1, a_2, \ldots a_s \quad \ldots\ldots\ldots\

It is to be noticed that S_a^2 is entirely independent of $a_{r+1}, a_{r+2}, \ldots a_s$. Thus we may write

$$p(x_1, x_2, \ldots x_n) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-n\frac{S_a^2+S_0^2}{2\sigma^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(49).$$

Now introduce new variates, say,

$$E_1, E_2, \ldots E_s, y_{s+1}, y_{s+2}, \ldots y_n \ldots\ldots\ldots\ldots\ldots\ldots(50),$$

connected with the x's by means of the formulae

$$x_i = m_i + c_{1i}E_1 + c_{2i}E_2 + \ldots + c_{si}E_s + c_{s+1,i}y_{s+1} + \ldots + c_{ni}y_n \ldots\ldots(51),$$

where c_{ki} for $k \leqslant s$ are the given coefficients of the formulae (4) and for $k > s$ are so chosen that

$$\sum_{i=1}^{n} c_{ki}c_{ji} = 0 \qquad \text{if } k \neq j \text{ and } j > s \quad \ldots\ldots\ldots\ldots\ldots\ldots(52),$$

and

$$\sum_{i=1}^{n} c_{ji}^2 = 1 \qquad \text{for } j > s \quad \ldots\ldots\ldots\ldots\ldots\ldots(53).$$

It is easily seen that it is always possible to find coefficients $c_{s+k,i}$ satisfying the above conditions, provided the matrix (5) is not singular.

In order to obtain the simultaneous frequency function of the new variables (50), we have to substitute (51) into (49) and to multiply the result into the Jacobian of the transformation. This Jacobian, say G, is certainly not zero. In fact it is easily seen that owing to the special choice of $c_{s+k,i}$, satisfying (52) and (53), the square G^2 is equal to the square of the matrix (5) and thus is positive, not zero.

Owing to (4) the equation (51) may be written in the form

$$x_i = c_{1i}(E_1 + a_1^0) + c_{2i}(E_2 + a_2^0) + \ldots + c_{si}(E_s + a_s^0) + c_{s+1,i}y_{s+1} + \ldots + c_{ni}y_n \quad \ldots\ldots(54).$$

Multiplying each of these equations into c_{ji} $(j = 1, 2, \ldots s)$ and summing, we see that the terms in y's are cancelling, owing to (52), and further that the sums $E_k + a_k$ are satisfying the same linear equations in terms of the x's as the a's, namely the equations (36). We thus have the identities

$$E_k = a_k - a_k^0 \qquad \text{for } k = 1, 2, \ldots r \quad \ldots\ldots\ldots\ldots\ldots\ldots(55),$$

and

$$E_k = a_k - a_k \qquad \text{for } k = r+1, r+2, \ldots s \ldots\ldots\ldots\ldots\ldots\ldots(56).$$

Using this we have

$$x_i - c_{1i}a_1 - c_{2i}a_2 - \ldots - c_{si}a_s = c_{s+1,i}y_{s+1} + \ldots + c_{ni}y_n \quad \ldots\ldots\ldots(57).$$

Squaring (57) and summing we get, owing to (48), (52) and (53),

$$nS_a^2 = \sum_{k=s+1}^{n} y_k^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58),$$

while

$$nS_0^2 = \sum_{i=1}^{n} (c_{1i}E_1 + c_{2i}E_2 + \ldots + c_{si}E_s)^2 \quad \ldots\ldots\ldots\ldots\ldots(59).$$

If we substitute (51) into (34) and multiply the result by G, we get

$$p(E_1, E_2, \dots E_s, y_{s+1}, y_{s+2}, \dots y_n) = \text{const.} \times e^{-n\frac{S_a^2+S_0^2}{2\sigma^2}} \quad \dots\dots(60),$$

and it is easily seen that, owing to the fact that S_a^2 is dependent only upon y and S_0^2 only upon E's, the variates y_i are independent of the E's, and besides are independent among themselves. The expression nS_a^2 being a sum of $n-s$ independent normal variates, it is known that the distribution of S_a follows the law

$$p(S_a) = \text{const.} \times S_a^{n-s-1} e^{-\frac{nS_a^2}{2\sigma^2}} \quad \dots\dots(61).$$

It follows that the simultaneous distribution of S_a, E_1, E_2, ... E_s is given by

$$p(S_a, E_1, E_2, \dots E_s) = \text{const.} \times S_a^{n-s-1} e^{-n\frac{S_a^2+S_0^2}{2\sigma^2}} \quad \dots\dots(62).$$

Further on we shall use this result.

Our problem being to find the distribution of S_b/S_a, we have now to consider the process of calculating S_b.

I have already pointed out that nS_a^2 is entirely independent of the a's. Thus when calculating the minimum value of the sum χ^2 in (46) with regard to the arbitrary parameters $a_{r+1}, a_{r+2}, \dots a_s$, we are really trying to minimise the sum of squares nS_0^2 in (47), where the a's and the a_i^0's are to be considered constant and the a_{r+k}'s are arbitrary. Thus S_b^2 will be the minimum of S_0^2. It will be a positive definite quadratic form in r variates $E_1, E_2, \dots E_r$. As these have been found to be independent of the y's and thus of S_a, S_b is also independent of S_a, and, as is easily seen, follows the distribution

$$p(S_b) = \text{const.} \times S_b^{r-1} e^{-\frac{nS_b^2}{2\sigma^2}} \quad \dots\dots(63).$$

Therefore the simultaneous distribution of S_a and S_b is given by the product

$$p(S_a, S_b) = \text{const.} \times S_a^{n-s-1} S_b^{r-1} e^{-n\frac{S_a^2+S_b^2}{2\sigma^2}} \quad \dots\dots(64).$$

Introducing instead of S_b the new variate

$$S_b = S_a Z \quad \dots\dots(65),$$

we get

$$p(S_a, Z) = \text{const.} \times S_a^{n-(s-r)-1} e^{-nS_a^2\frac{1+Z^2}{2\sigma^2}} Z^{r-1} \quad \dots\dots(66).$$

Now on integrating with regard to S_a within the limits 0 and ∞, we get

$$p(Z) = \text{const.} \times Z^{r-1}(1+Z^2)^{-\frac{n-(s-r)}{2}} \quad \dots\dots(67),$$

and the problem may be considered as solved. The tests obtained are those invented and applied in many cases by R. A. Fisher.

The expressions

$$\frac{nS_a^2}{n-s}, \quad \frac{nS_b^2}{r} \quad \dots\dots(68)$$

are considered by him as independent estimates of the common variance.

The method of testing is reduced to the calculation of

$$z = \tfrac{1}{2} \log \frac{S_b^2 (n-s)}{S_a^2 r} = \log Z + \tfrac{1}{2} \log \frac{n-s}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (69),$$

and to referring to the table of values of z corresponding to the "confidence" coefficient $\epsilon = \cdot 05$ and $\epsilon = \cdot 01$, given by R. A. Fisher, where n_1 and n_2 correspond to my r and $n-s$ respectively.

Another method of course would be to substitute in (67), say,

$$u = (1 + Z^2)^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (70),$$

and in obtaining $$p(u) = \frac{1}{B\left(\frac{n-s}{2}, \frac{r}{2}\right)} u^{\frac{n-s}{2}-1} (1-u)^{\frac{r}{2}-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (71),$$

and in testing whether $$(1 + Z^2)^{-1} \leqslant u_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (72),$$

where u_0 is such as to have $$\int_0^{u_0} p(u)\, du = \epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots (73).$$

The value of u_0 may be found from the Incomplete Beta-Function Tables*.

Of course if $r = 1$, then the distribution of Z reduces itself to that of "Student's" t.

It is interesting that the tests of R. A. Fisher are consequences of the principle of likelihood, stated above.

I think it will be useful to finish this section by summarising the mechanical process of testing any hypothesis of the type H and in illustrating it by one example.

The steps in testing a hypothesis H are as follows:

1. State the set of admissible hypotheses Ω and test whether each of them is describing the distribution of any variate x_i $(i = 1, 2, \ldots n)$ by means of the formula

$$p(x_i) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x_i - m_i)^2}{2\sigma^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (74),$$

where σ is independent of i, and m_i may be represented by a linear function of certain $s < n$ unknown parameters a.

2. Express the hypothesis H tested by means of independent linear equations (let r be their number) with regard to the parameters a. If this is possible, then:

3. Calculate the minimum value of

$$\chi^2 = \sum_{i=1}^{n} (x_i - m_i)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (75)$$

which it may have for any set of values of the parameters a, corresponding to admissible hypotheses contained in Ω. This minimum value will be nS_a^2. Divide the minimum obtained by the number of degrees of freedom, equal to n, the

* K. Pearson, *Tables of the Incomplete Beta Function*. *Biometrika* Office, 1934.

total number of observations, minus s, the number of parameters a. The ratio will be

$$\frac{nS_a^2}{n-s} \quad \text{.....(76).}$$

4. Calculate the minimum value that χ^2 may have for values of the parameters a, satisfying the r linear equations, expressing the hypothesis to be tested. The minimum will be $n(S_a^2+S_b^2)$. Subtract nS_a^2, obtained previously, and divide by the number of degrees of freedom, equal to r, the number of independent equations expressing H. The result will be

$$\frac{nS_b^2}{r} \quad \text{.....(77).}$$

5. Calculate z, according to (69), and refer to Fisher's tables.

5. EXAMPLE: THE PROBLEM PROPOSED BY K. IWASZKIEWICZ.

Obviously the above five steps must be divided into two parts. Everything up to and including the determination of formulae giving S_a^2 and S_b^2 may be regarded as theoretical work. The practical test will then consist of numerical calculations leading to z, for which different formulae simplifying the arithmetic may be given.

The example I shall consider is that of the problem proposed by Miss K. Iwaszkiewicz, to test the hypothesis H whether the regression coefficients of x on y in k different populations, from which we have samples, are equal. Suppose that the sample, Σ_i, from the ith population, π_i, consists of n_i individuals, for which the values of x and y are respectively

$$\left.\begin{matrix} x_{i1},\ x_{i2},\ \dots\ x_{in_i} \\ y_{i1},\ y_{i2},\ \dots\ y_{in_i} \end{matrix}\right\} \quad \text{.....(78).}$$

We consider each population π_i as being subdivided into partial populations, say π_{iy}, according to the values of y, appropriate to the individuals of π_{iy}. Each individual of the sample Σ_i may be considered as being randomly drawn from a separate population π_{iy}.

We consider as given that the regression of x on y in each population π_i is linear. This is equivalent to the assumption that the mean, m_{iy}, of the population π_{iy} is equal to

$$m_{iy} = a_{i1} + a_{i2}y \quad \text{.....(79),}$$

where a_{i1} and a_{i2} are some unknown parameters, the number of which is $s = 2k$.

If there is evidence that the variances of the x's within single partial populations π_{iy} have equal values σ^2, we may proceed further.

The hypothesis to be tested may be expressed in

$$r = k - 1 \quad \text{.....(80)}$$

independent linear equations in terms of the parameters a, namely,

$$a_{i2} - a_{k2} = 0 \qquad (i = 1, 2, \dots k-1) \text{.....(81).}$$

We consider the sum of squares, χ^2, which may be written

$$\chi^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - a_{i1} - a_{i2} y_{ij})^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (82).$$

The process of minimising this sum is easy, and we find

$$n S_a^2 = \sum_{i=1}^{k} n_i \sigma_{x_i}^2 (1 - r_i)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (83),$$

where σ_{x_i} and r_i mean respectively the standard deviation of x's and the coefficient of correlation between x and y in the sample Σ_i.

Now we shall have to assume that the parameters a satisfy the equations by which the hypothesis H is expressed. Denote by a_2 the common value of a_{i2}. Obviously $nS_a^2 + nS_b^2$ will be the minimum of the sum, say,

$$\chi_1^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - a_{i1} - a_2 y_{ij})^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (84),$$

with regard to all possible values of the parameters a_2 and a_{i1}, for $i = 1, 2, \ldots k$. After easy algebra we have

$$n(S_a^2 + S_b^2) = \sum_{i=1}^{k} n_i \sigma_{x_i}^2 - \frac{\left(\sum_{i=1}^{k} n_i r_i \sigma_{x_i} \sigma_{y_i} \right)^2}{\sum_{i=1}^{k} n_i \sigma_{y_i}^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (85),$$

where σ_{y_i} means the standard deviation of y's in Σ_i. Some further easy algebra gives us the result

$$n S_b^2 = \frac{\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} n_i n_j \sigma_{y_i}^2 \sigma_{y_j}^2 (a_i - a_j)^2}{\sum_{i=1}^{k} n_i \sigma_{y_i}^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (86),$$

where

$$a_i = r_i \frac{\sigma_{x_i}}{\sigma_{y_i}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (87)$$

means the regression coefficient of x on y in the sample Σ_i. The above formula is certainly not the most convenient one for calculation, but it shows the connection of the test with the differences in the regressions within single samples. The formulae which are the most convenient for calculation are, as usual, the less elegant ones, and may be easily deduced directly from the process of minimising (82) and (84).

6. The Method of Testing the Hypothesis H, based on the Best Critical Regions.

We shall now treat the same problem from the point of view of the theory of Best Critical Regions*. If we consider the data of the observations

$$x_1, x_2, \ldots x_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (88)$$

as co-ordinates of a point Σ (the "sample point") in the n-dimensioned space (the sample space W), then any method of testing the hypothesis H consists in a rule

* Neyman and Pearson, "On the Problem of the most efficient Tests of Statistical Hypotheses." (See reference 9.)

of rejecting H, whenever the sample point lies within a certain region, say w, in the sample space. The region w will be called the critical region; the supplementary region $W-w$ the region of acceptance. If the hypothesis H is a composite one to be tested, thus leaving one or more parameters determining the simultaneous distribution of the x's unspecified, then the appropriate critical region w should be "similar" to the sample space with regard to these unspecified parameters. I shall now explain this concept having in view the particular hypothesis H, considered in previous sections.

The hypothesis H does not specify the parameters

$$\sigma, a_{r+1}, a_{r+2}, \dots a_p \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(89).$$

We shall say that a region w is similar to W with regard to these parameters and that it has the power ϵ, if the probability, say $P\{(\Sigma \epsilon w)|H\}$*, determined by any simple hypothesis belonging to H, that the sample point Σ will fall within w is equal to ϵ, *whatever be the values of the parameters* (89), unspecified by H.

In order to define the best critical region, we have to consider an alternative simple hypothesis, say h_1. We shall say that w_0 is the best critical region with regard to h_1, having the power ϵ, if

(1) w_0 is similar to W with regard to (89) and has the power ϵ, and

(2) if the probability determined by the alternative hypothesis h_1 that the sample point Σ will be contained in w_0 is larger than (or at least equal to) the same probability corresponding to any other region, w, satisfying (1).

We see that the process of finding the best critical regions must be preceded by finding the regions similar to W.

Neyman and Pearson gave the method of determining the most general regions similar to the sample space, the application of which is possible under the following conditions:

(a) The simultaneous probability function $p\{(x_1, x_2, \dots x_n)|H\}$ should possess the derivatives of all orders with regard to all parameters unspecified by H for every set of values of $x_1, x_2, \dots x_n$.

(b) If a is a parameter unspecified by the hypothesis, then denoting

$$\phi_a = \frac{\partial \log p}{\partial a} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(90),$$

we should have, say,

$$\phi_a' = \frac{\partial \phi_a}{\partial a} = A + B\phi_a \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(91),$$

where A and B are independent of the x's.

(c) The hypothesis H does not specify the values of $s-r+1$ parameters. Assume that these parameters are written in any order and form a series, say,

$$a_1, a_2, \dots a_{s-r+1} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(92).$$

* The notation used in this formula, $\Sigma \epsilon w$, means: the point Σ is an element of w. It is generally used in the theory of sets. The letter ϵ standing here for "element" should not be confused with ϵ denoting the power of the region w.

Denote by ϕ_i the function defined as in (90),

$$\phi_i = \frac{\partial \log p}{\partial a_i} \quad \text{.....(93),}$$

and consider the hypersurfaces

$$\phi_j = C_j = \text{const.} \qquad (j = 1, 2, \ldots s-r+1) \ldots (94).$$

Fix the values of the parameters a and C and consider the intersection of the $i-1$ hypersurfaces (94) corresponding to $j = 1, 2, \ldots i-1$. This intersection will be denoted by $S(a_i; C_1, C_2, \ldots C_{i-1})$. Keep the values of a's constant and change the C's in an arbitrary way. Doing so we shall obtain a family of intersections $S(a_i; C_1, C_2, \ldots C_{i-1})$. This family will be denoted by $F(a_i)$. The condition (c) requires that it should be possible to find such an order of parameters unspecified by the hypothesis tested, that each family $F(a_i)$, for $i = 2, 3, \ldots s-r+1$, should be independent of a_i. This condition means that given any two different values of a_i, say a_i' and a_i'', any surface $S(a_i'; C_1, \ldots C_{i-1})$ included in the family $F(a_i')$ should be included also in $F(a_i'')$, corresponding perhaps to some values of the C's different from those to which it corresponded in $F(a_i')$.

If the above conditions (a), (b) and (c) are satisfied, then the method of finding the similar regions may be applied and the best critical region is constructed in the following way.

Consider the intersection of $s-r+1$ surfaces (94) corresponding to fixed values of parameters (compatibie with the hypothesis tested H) and to some fixed values of the constants C. This will be denoted by $W(C_1, C_2, \ldots C_{s-r+1})$. Denote by $w_0(C_1, C_2, \ldots C_{s-r+1})$ the part of the hypersurface $W(C_1, C_2, \ldots C_{s-r+1})$ defined by the inequality

$$p\{(x_1, x_2, \ldots x_n) | h_1\} \geqslant p\{(x_1, x_2, \ldots x_n) | H\}\, k(C_1, C_2, \ldots C_{s-r+1}) \quad \ldots(95),$$

where $k(C_1, C_2, \ldots C_{s-r+1})$ must be found for any system of the C's so as to satisfy the condition

$$\iint \ldots \int_{w_0(C_1, \ldots C_{s-r+1})} p\{(x_1, \ldots x_n) | H\}\, dw(C_1, \ldots C_{s-r+1})$$
$$= \epsilon \iint \ldots \int_{W(C_1, \ldots C_{s-r+1})} p\{(x_1, \ldots x_n) | H\}\, dW(C_1, \ldots C_{s-r+1}) \quad \text{.....(96).}$$

The best critical region w_0 is then built up of regions $w_0(C_1, \ldots C_{s-r+1})$ corresponding to all possible values of constants $C_1, C_2, \ldots C_{s-r+1}$.

We have now to consider whether the probability function $p\{(x_1, \ldots x_n) | H\}$ satisfies the conditions (a), (b) and (c).

Ad (a). The condition (a) is obviously satisfied.

Ad (b). Turning to our previous notation concerning the parameters, we shall have

$$\log p\{(x_1, x_2, \ldots x_n) | H\}$$
$$= -n \log \sqrt{2\pi} - n \log \sigma - \frac{\sum_{i=1}^{n} (x_i - c_{1i}a_1 - \ldots - c_{ri}a_r - c_{r+1,i}a_{r+1} - \ldots - c_{si}a_s)^2}{2\sigma^2}$$
$$\text{.....(97),}$$

$$\phi_\sigma = \frac{\partial \log p}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} \left(x_i - \sum_{j=1}^{s} c_{ji} a_j\right)^2 \quad \text{.....(98),}$$

$$\phi_\sigma' = \frac{\partial \phi_\sigma}{\partial \sigma} = \frac{n}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=1}^{n} \left(x_i - \sum_{j=1}^{s} c_{ji} a_j\right)^2$$

$$= -\frac{2n}{\sigma^2} - \frac{3}{\sigma} \phi_\sigma \quad \text{.....(99).}$$

Thus ϕ_σ satisfies the equation (91). Consider now, say,

$$\phi_{aj} = \frac{\partial \log p}{\partial a_j} = \frac{1}{\sigma^2} \sum_{i=1}^{n} \left(x_i - \sum_{t=1}^{s} c_{ti} a_t\right) c_{ji} \quad \text{.....(100).}$$

Obviously $$\phi_{aj}' = \frac{\partial \phi_{aj}}{\partial a_j} = -\frac{1}{\sigma^2} \sum_{i=1}^{n} c_{ji}^2 \quad \text{.....(101),}$$

and satisfies (91) as it is independent of the x's, and B may be assumed to be zero.

Ad (c). The hypothesis (c) is also satisfied. To show this we write the parameters, which are not specified by the hypothesis H, in the following order:

$$a_1 = a_{r+1};\ a_2 = a_{r+2};\ \dots\ a_{s-r} = a_s;\ a_{s-r+1} = \sigma \quad \text{.....(102).}$$

For every $2 \leqslant i \leqslant s-r$ the hypersurface $S(a_i;\ C_1, C_2, \dots C_{i-1})$ is the intersection of the hyperplanes, the equations of which are obtained by equating (100) to arbitrary constants, thus equivalent to the following equations:

$$\sum_{i=1}^{n} c_{ji} x_i = C_j \qquad (j = r+1, r+2, \dots r+i-1) \dots (103).$$

These equations being independent of $a_i = a_{r+1}$, so is the family $F(a_i)$. Now put $i = s-r+1$. The hypersurface $S(a_{s-r+1};\ C_1, C_2, \dots C_{s-r})$ is now the intersection of hyperplanes (103) for $j = r+1, r+2, \dots s$, which are independent of $a_{s-r+1} = \sigma$. This means that $F(a_{s-r+1})$ is independent of $a_{s-r+1} = \sigma$.

Thus all conditions which are sufficient for the applicability of the Neyman-Pearson method are satisfied and we may proceed to the construction of the best critical region w_0.

Consider then hypersurfaces $W(C_1, \dots C_{s-r+1})$, formed by the intersection of hyperplanes (103) for $j = 1, 2, \dots s-r$ and of the hypersphere

$$\phi_\sigma = C_{s-r+1} \quad \text{.....(104).}$$

According to the general theory the best critical region with regard to the alternative hypothesis h_1 consists of parts $w(C_1, \dots C_{s-r+1})$ of the hypersurfaces $W(C_1, \dots C_{s-r+1})$ determined by the inequality

$$p\{(x_1, x_2, \dots x_n) | h_1\} \geqslant p\{(x_1, x_2, \dots x_n) | H\}\, k(C_1, C_2, \dots C_{s-r+1}) \dots (105),$$

where $k(C_1, C_2, \dots C_{s-r+1})$ must be chosen so as to satisfy the equation

$$\iint \dots \int_{w(C_1, \dots C_{s-r+1})} p\{(x_1, \dots x_n) | H\}\, dw(C_1, \dots C_{s-r+1})$$

$$= \epsilon \iiint_{W(C_1, \dots C_{s-r+1})} p\{(x_1, \dots x_n) | H\}\, dW(C_1, \dots C_{s-r+1}) \quad \text{.....(106)}$$

on every hypersurface $W(C_1, \dots C_{s-r+1})$, corresponding to every possible system of values of the constants $C_1, \dots C_{s-r+1}$.

The solution is somewhat simpler if we introduce new variables instead of (88).

Denote by

$$a_1^0, a_2^0, \dots a_r^0, a_{r+1}, \dots a_s, \sigma \quad \dots\dots\dots\dots(107)$$

the values of parameters corresponding to the hypothesis H, and by

$$a_1', a_2', \dots a_r', a_{r+1}, \dots a_s, \sigma \quad \dots\dots\dots\dots(108)$$

those corresponding to the alternative h_1. The best critical region and the integral over this region from $p\{(x_1, \dots x_n) | H\}$ being independent of the values of the last $s-r+1$ parameters (107), we may assume that their values are the same as in (108).

Now instead of the x's we introduce new variates

$$E_1, E_2, \dots E_s, y_{s+1}, y_{s+2}, \dots y_n \dots\dots\dots\dots(109),$$

connected with the x's by means of the formulae (54). Instead of considering the probability function $p\{(x_1, \dots x_n) | H\}$ we shall have to consider the function (60), namely,

$$p\{(E_1, E_2, \dots E_s, y_{s+1}, y_{s+2}, \dots y_n) | H\} = \text{const. } e^{-n\frac{S_a^2 + S_0^2}{2\sigma^2}} \dots(60\ bis),$$

where

$$nS_a^2 = \sum_{k=s+1}^{n} y_k^2 \dots\dots\dots\dots(58\ bis),$$

and

$$nS_0^2 = \sum_{i=1}^{n} (c_{1i}E_1 + c_{2i}E_2 + \dots c_{si}E_s)^2 \quad \dots\dots\dots\dots(59\ bis).$$

Having regard to (55) and (56) we conclude that the transformation used will lead to the following formula:

$$p\{(E_1, E_2, \dots E_s, y_{s+1}, y_{s+2}, \dots y_n) | h_1\} = \text{const.} \times e^{-n\frac{S_a^2 + S_1^2}{2\sigma^2}} \dots(110),$$

where S_a^2 is the same as in (60 *bis*) but where, instead of S_0^2 given by (59 *bis*), we shall have S_1^2 determined by the formula

$$nS_1^2 = \sum_{i=1}^{n} (c_{1i}E_1' + c_{2i}E_2' + \dots + c_{ri}E_r' + c_{r+1}E_{r+1} + \dots + c_{si}E_s)^2 \dots(111),$$

if

$$E_k' = E_k - \epsilon_k \qquad (k = 1, 2, \dots r) \dots\dots\dots(112)$$

and

$$\epsilon_k = a_k' - a_k^0 \qquad (k = 1, 2, \dots r) \dots\dots\dots(113).$$

We shall make another linear transformation, introducing instead of $E_{r+1}, E_{r+2}, \dots E_s$ new variables, say $\psi_{r+1}, \psi_{r+2}, \dots \psi_s$, which will transform nS_0^2 into

$$nS_0^2 = \sum_{ij=1}^{r} R_{ij}E_iE_j + \sum_{j=r+1}^{s} \psi_j^2 \quad \dots\dots\dots\dots(114).$$

The same transformation will give us

$$nS_1^2 = \sum_{ij=1}^{r} R_{ij}E_i'E_j' + \sum_{j=r+1}^{s} \psi_j^2 \quad \dots\dots\dots\dots(115).$$

It will be seen that the quadratic form

$$\sum_{ij=1}^{r} R_{ij} E_i E_j = nS_b^2 \quad \text{.....(116)}$$

is positive and definite and represents the minimum value of nS_b^2 with regard to all possible systems of values of $a_{r+1}, a_{r+2}, \ldots a_s$.

The coefficients R_{ij} may be obtained from the coefficients c_{ij} as follows*. Denote by

$$g_{ij} = \sum_{t=1}^{n} c_{it} c_{jt} = g_{ji} \quad \text{.....(117)}$$

and consider the determinant, say,

$$G = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1s} \\ g_{21} & g_{22} & \cdots & g_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ g_{s1} & g_{s2} & \cdots & g_{ss} \end{vmatrix} \quad \text{.....(118).}$$

Denote further by G_{ij}' the minor of G, corresponding to g_{ij}, and let

$$G_{ij} = \frac{G_{ij}'}{G} \quad \text{.....(119).}$$

* This is an application of the following general theorem concerning the linear transformations, which we shall have to use again below.

If

$$\sum_{ij=1}^{s} g_{ij} x_i x_j \quad \text{.....(1)}$$

is a positive definite form and the functions

$$y_p = b_{p1} x_1 + b_{p2} x_2 + \ldots + b_{ps} x_s \qquad (p = 1, 2, \ldots r < s) \quad \text{.....(2)}$$

are independent, then it is always possible to find another $s - r$ linear function, say,

$$y_q = b_{q1} x_1 + b_{q2} x_2 + \ldots + b_{qs} x_s \qquad (q = r+1, r+2, \ldots s) \quad \text{.....(3),}$$

such that together with (3) they form an independent system. Moreover, if the system of equations (2) and (3) is solved with regard to the x's and the solutions are substituted into (1) then we get

$$\sum_{ij=1}^{s} g_{ij} x_i x_j = \sum_{ij=1}^{r} T_{ij} y_i y_j + \sum_{t=r+1}^{s} y_t^2 \quad \text{.....(4).}$$

The coefficients T_{ij} in (4) may be calculated from the coefficients g_{ij} in (1) and b_{pi} in (2). Consider the determinant

$$G = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1s} \\ g_{21} & g_{22} & \cdots & g_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ g_{s1} & g_{s2} & \cdots & g_{ss} \end{vmatrix} \quad \text{.....(5),}$$

and denote by G_{ij} the elements of its reciprocal, say, G^{-1}. That is to say, G_{ij} will be equal to the minor of G corresponding to g_{ij}, divided into G. Consider further the determinant, say,

$$T = \begin{vmatrix} t_{11} & t_{12} & \cdots & t_{1r} \\ t_{21} & t_{22} & \cdots & t_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ t_{r1} & t_{r2} & \cdots & t_{rr} \end{vmatrix} \quad \text{.....(6),}$$

where

$$t_{pq} = \sum_{ij=1}^{s} b_{pi} b_{qi} G_{ij} \quad \text{.....(7).}$$

Now T_{pq} will be the element of the determinant T^{-1}, reciprocal to T. In other words T_{pq} is equal to the minor of T, corresponding to t_{pq}, divided into T.

Consider finally the determinant, say,

$$R = \begin{vmatrix} G_{11} & G_{12} & \dots & G_{1r} \\ G_{21} & G_{22} & \dots & G_{2r} \\ \dots & \dots & \dots & \dots \\ G_{r1} & G_{r2} & \dots & G_{rr} \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(120),$$

and let R_{ij}' be its minor corresponding to G_{ij}. Then we shall have

$$R_{ij} = \frac{R_{ij}'}{R} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(121).$$

The probability functions of the new variables

$$E_1, E_2, \dots E_r, \psi_{r+1}, \psi_{r+2}, \dots \psi_s, y_{s+1}, y_{s+2}, \dots y_n \quad \dots\dots\dots\dots(122),$$

determined by the two hypotheses H and h_1 will have the previous forms (60 *bis*) and (110) respectively, with the exception that the expressions of nS_0^2 and nS_1^2 will now be (114) and (115).

We shall now transform to new variables the equations (103) and (104). It is easily seen that the equations (103) are equivalent to the following ones:

$$\frac{\partial (nS_0^2)}{\partial a_j} = \text{const.} \qquad (j = r+1, r+2, \dots s) \dots(123);$$

where nS_0^2 should have the form (59 *bis*), and where

$$E_j = a_j - a_j \qquad (j = 1, 2, \dots s) \quad \dots\dots\dots\dots(124).$$

But we may also use the form (114). The relationships between the ψ's and the E's may be written

$$\psi_j = \sum_{t=1}^{s} b_{jt} E_t \qquad (j = r+1, r+2, \dots s) \quad \dots(125).$$

If we substitute these formulae into (114), we shall get identically (59 *bis*),

$$nS_0^2 = \sum_{ij=1}^{r} R_{ij} E_i E_j + \sum_{j=r+1}^{s} \left(\sum_{t=1}^{s} b_{jt} E_t \right)^2 \quad \dots\dots\dots\dots\dots(126).$$

The equations (103) transformed to the new variables may now be obtained by differentiating (126) with regard to a_j $(j = r+1, r+2, \dots s)$, and keeping in view (124) and (125). We have

$$\frac{\partial (nS_0^2)}{\partial a_j} = -2 \sum_{i=r+1}^{s} \psi_i b_{ij} \quad \text{for } j = r+1, r+2, \dots s \quad \dots\dots(127).$$

Thus the new form of the equations (103) will be

$$\sum_{i=r+1}^{s} b_{ij} \psi_i = C_j \qquad (j = r+1, r+2, \dots s) \dots\dots(128),$$

C_j being an arbitrary constant. The equations (128) depend only upon ψ's. They may be solved with regard to the ψ's and thus replaced by an equivalent system

$$\psi_{r+1} = D_{r+1}, \psi_{r+2} = D_{r+2}, \dots \psi_s = D_s \quad \dots\dots\dots\dots\dots(129),$$

where the D's are again arbitrary constants.

Turn now to equation (104). This is equivalent to

$$n(S_a^2 + S_0^2) = \text{const.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (130).$$

Owing to equations (114) it may be replaced by

$$n(S_a^2 + S_0^2) = \psi_0 = D_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (131),$$

D_0 being an arbitrary constant.

The problem of finding the best critical region is now reduced to that of determining the hypersurface, say $w(D_0, D_{r+1}, D_{r+2}, \ldots D_s)$, which is a part of the intersection of (129) and (131) on which

$$p\{(E_1, \ldots E_r, \psi_{r+1}, \ldots \psi_s, y_{s+1}, \ldots y_n)|h_1\}$$
$$\geqslant p\{(E_1, \ldots E_r, \psi_{r+1}, \ldots \psi_s, y_{s+1}, \ldots y_n)|H\}\, k(D_0, D_{r+1}, \ldots D_s) \quad \ldots\ldots (132),$$

k being subject to the condition of satisfying (106). Taking the logarithms of both sides of (132) we shall get

$$-\frac{1}{2\sigma^2}\left[nS_a^2 + \sum_{ij=1}^{r} R_{ij} E_i' E_j' + \sum_{t=r+1}^{s} \psi_t^2\right]$$
$$\geqslant -\frac{1}{2\sigma^2}\left[nS_a^2 + \sum_{ij=1}^{r} R_{ij} E_i E_j + \sum_{t=r+1}^{s} \psi_t^2\right] + k \quad \ldots\ldots\ldots (133),$$

and after some reduction

$$\sum_{ij=1}^{r} R_{ij}(E_i E_j - E_i' E_j') \geqslant k\sigma^2 = k_1 \quad \text{(say)} \quad \ldots\ldots\ldots\ldots\ldots (134).$$

Owing to the relation $E_i' = E_i - \epsilon_i$ ..... (135),

we have $E_i E_j - E_i' E_j' = \epsilon_i E_j + \epsilon_j E_i - \epsilon_i \epsilon_j$ ..... (136).

Thus (134) may be written in an equivalent form:

$$2\sum_{ij=1}^{r} R_{ij} \epsilon_i E_j - \sum_{ij=1}^{r} R_{ij} \epsilon_i \epsilon_j \geqslant k_1 \quad \ldots\ldots\ldots\ldots\ldots (137),$$

or

$$\xi_1 = \sum_{ij=1}^{r} R_{ij} \epsilon_i E_j \geqslant k_2 \quad \text{(say)} \quad \ldots\ldots\ldots\ldots\ldots (138).$$

The region $w(D_0, D_{r+1}, \ldots D_s)$ is the part of the intersection of (129) and (131), say $W(D_0, D_{r+1}, \ldots D_s)$, determined by the inequality (138), where k_2 should be so chosen that the integral of the probability function corresponding to H over $w(D_0, D_{r+1}, \ldots D_s)$ should be equal to ϵ multiplied into the integral of the same function over $W(D_0, D_{r+1}, \ldots D_s)$. It will be seen that the problem of determining $w(D_0, D_{r+1}, \ldots D_s)$ is that of finding such a value k_2, that the relative probability (determined by H) of (138), given (129) and (131), may be equal to ϵ:

$$P\{(\xi_1 \geqslant k_2)|(H), \quad (\psi_0 = D_0), \quad (\psi_{r+1} = D_{r+1}), \ldots (\psi_s = D_s)\} = \epsilon \quad \ldots\ldots (139).$$

This last problem would be easily solved if we could determine the simultaneous frequency function, say, $p(\xi_1, \psi_0, \psi_{r+1}, \ldots \psi_s)$ of variates $\xi_1, \psi_0, \psi_{r+1}, \ldots \psi_s$.

As all these variates depend only upon

$$S_a, E_1, E_2, \ldots E_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (140),$$

we may start with the respective frequency function given by the formula (62),

$$p(S_a, E_1, E_2, \ldots E_s) = \text{const.} \times S_a^{n-s-1} e^{-n\frac{S_a^2 + S_0^2}{2\sigma^2}} \quad \ldots\ldots\ldots (141),$$

where nS_0^2 has the form (59 *bis*). Introducing $\psi_{r+1}, \psi_{r+2}, \dots \psi_s$ as new variables, by means of the formulae (125) we get $p(S_a, E_1, \dots E_r, \psi_{r+1}, \dots \psi_s)$, having the same form as (141) except that nS_0^2 has now the expression (114),

$$p(S_a, E_1, \dots E_r, \psi_{r+1}, \dots \psi_s) = \text{const.} \times S_a^{n-s-1} e^{-n\frac{S_a^2+S_b^2+S_c^2}{2\sigma^2}} \quad \dots(142),$$

where

$$nS_c^2 = \sum_{t=r+1}^{s} \psi_t^2 \quad \dots(143).$$

Our next step will consist in changing the system of variables,

$$S_a, E_1, \dots E_r, \psi_{r+1}, \dots \psi_s,$$

into the new one,

$$\psi_0, \psi_1, \psi_2, \dots \psi_s \quad \dots(144),$$

by means of the formulae

$$\left.\begin{aligned} \psi_i &= \sum_{j=1}^{r} R_{ij} E_j \quad \text{for} \quad i = 1, 2, \dots r \\ \psi_i &= \psi_{i'} \quad \text{for} \quad i \text{ or } i' = r+1, r+2, \dots s \\ \psi_0 &= nS_a^2 + nS_b^2 \end{aligned}\right\} \quad \dots(145).$$

It is easily seen that

$$nS_b^2 = \sum_{pq=1}^{r} G_{pq} \psi_p \psi_q \quad \dots(146).$$

The Jacobian of the transformation being

$$\frac{\partial(S_a, E_1, \dots E_r, \psi_{r+1}, \dots \psi_s)}{\partial(\psi_1, \dots \psi_s, \psi_0)} = \left[\frac{\partial(\psi_1, \dots \psi_s, \psi_0)}{\partial(S_a, E_1, \dots E_r, \psi_{r+1}, \dots \psi_s)}\right]^{-1} = (2n\, S_a\, R)^{-1} \quad \dots(147),$$

we find

$$p(\psi_0, \psi_1, \dots \psi_s) = \text{const.} \times (\psi_0 - nS_b^2)^{\frac{n-s-2}{2}} e^{-\frac{\psi_0 + nS_c^2}{2\sigma^2}} \quad \dots(148).$$

The last step in this series of transformations will consist in introducing ξ_1, a new variable, and also in representing nS_b^2 in (146) as a sum of squares. This will be done by applying again the theorem of algebra I have given in the footnote on p. 177.

It will be seen that substituting (145) in (138) we get

$$\xi_1 = \sum_{i=1}^{r} \epsilon_i \psi_i \quad \dots(149).$$

Applying the theorem just mentioned we may add to this equation another $r-1$ of them,

$$\xi_i = \sum_{i=1}^{r} e_{ij} \psi_j \qquad (i = 2, 3, \dots r) \quad \dots(150),$$

such that they will form a system of independent equations, and if used to transform (146) to the new variables $\xi_1, \xi_2, \dots \xi_r$ this will give, say,

$$nS_b^2 = Z_{11}^2 \xi_1^2 + \sum_{t=2}^{r} \xi_t^2 \quad \dots(151),$$

where

$$Z_{11}^{-2} = \sum_{ij=1}^{r} R_{ij} \epsilon_i \epsilon_j \quad \dots(152).$$

The Jacobian of the transformation of $\psi_0, \psi_1, \dots \psi_s$ into the new system

$$\xi_1, \xi_2, \dots \xi_r, \psi_{r+1}, \dots \psi_s, \psi_0$$

being obviously constant and different from zero, we get

$$p(\xi_1, \xi_2, \dots \xi_r, \psi_0, \psi_{r+1}, \dots \psi_s) = \text{const.} \times \left(\psi_0 - Z_{11}^2 \xi_1^2 - \sum_{t=2}^{r} \xi_t^2\right)^{\frac{n-s-2}{2}} e^{-\frac{\psi_0 + nS_c^2}{2\sigma^2}} \quad \dots\dots(153).$$

The required probability function of $\xi_1, \psi_0, \psi_1, \dots \psi_s$ is now easily found by integrating for $\xi_i\ (i=2, 3, \dots r)$ within the extreme limits of these variates, compatible with the fixed values of the others. The extreme limits for ξ_r will be, say,

$$-L_r \leqslant \xi_r \leqslant +L_r \quad \dots\dots(154),$$

if

$$L_r = \sqrt{\psi_0 - Z_{11}^2 \xi_1^2 - \sum_{t=2}^{r-1} \xi_t^2} \quad \dots\dots(155).$$

Putting $\xi_r = L_r u$ we shall get

$$\int_{-L_r}^{+L_r} p(\xi_1, \dots \xi_r, \psi_0, \psi_1, \dots \psi_s)\, d\xi_r = \text{const.}$$
$$\times \left(\psi_0 - Z_{11}^2 \xi_1^2 - \sum_{t=2}^{r-1} \xi_t^2\right)^{\frac{n-s-1}{2}} e^{-\frac{\psi_0 + nS_c^2}{2\sigma^2}} \int_{-1}^{+1} (1-u^2)^{\frac{n-s-2}{2}}\, du \quad \dots\dots(156),$$

where the integral with regard to u may be included in the constant factor. Obviously the process of integration may be repeated till we are left with

$$p(\xi_1, \psi_0, \psi_1, \dots \psi_s) = \text{const.} \times (\psi_0 - Z_{11}^2 \xi_1^2)^{\frac{n-s+r-3}{2}} e^{-\frac{\psi_0 + nS_c^2}{2\sigma^2}} \quad \dots(157).$$

Denoting the left-hand side of (139) for short by $P(k_2)$, we shall now get

$$P(k_2) = \frac{\int_{k_2}^{L} p(\xi_1, \psi_0, \psi_1, \dots \psi_s)\, d\xi_1}{\int_{-L}^{+L} p(\xi_1, \psi_0, \psi_1, \dots \psi_s)\, d\xi_1} \quad \dots\dots(158),$$

where $-L$ and $+L$ are the extreme limits of variation of ξ_1 compatible with the fixed value of ψ_0, namely,

$$L = \frac{\sqrt{\psi_0}}{Z_{11}} \quad \dots\dots(159).$$

Equating (158) to ϵ and performing the easy algebra, we get

$$\int_{k_2}^{L} (\psi_0 - Z_{11}^2 \xi_1^2)^{\frac{n-s+r-3}{2}}\, d\xi_1 = \epsilon \int_{-L}^{+L} (\psi_0 - Z_{11}^2 \xi_1^2)^{\frac{n-s+r-3}{2}}\, d\xi_1 \quad \dots(160).$$

Putting

$$\xi_1 = Lv, \quad k_2 = Lv_0 \quad \dots\dots(161),$$

v being a new variable, we reduce equation (160) to the following:

$$\int_{v_0}^{1} (1-v^2)^{\frac{n-s+r-3}{2}}\, dv = \epsilon \int_{-1}^{+1} (1-v^2)^{\frac{n-s+r-3}{2}}\, dv = \epsilon B\left(\frac{1}{2}, \frac{n-s+r-1}{2}\right)$$
$$\dots\dots(162).$$

From this equation v_0 may be found, and it appears to be an absolute constant, depending only upon ϵ. Taking into account (161), (159), (152) and the last of formulae (145) we get

$$k_2 = v_0 \sqrt{\sum_{ij=1}^{r} R_{ij}\epsilon_i\epsilon_j}\sqrt{nS_a^2 + nS_b^2} \quad \text{.....(163).}$$

The inequality (138) determining the region $w(D_0, D_{r+1}, \dots D_s)$ may be written in the form

$$\frac{1}{\sqrt{nS_a^2 + nS_b^2}} \frac{\sum_{ij=1}^{r} R_{ij}\epsilon_i E_j}{\sqrt{\sum_{ij=1}^{r} R_{ij}\epsilon_i\epsilon_j}} \geqslant v_0 \quad \text{.....(164).}$$

As this inequality is independent of the values $D_0, D_{r+1}, \dots D_s$, we see that it determines the best critical region w_0, provided v_0 is found to satisfy (162).

7. Is there a Single Best Critical Region with Regard to the Whole Class of Alternative Hypotheses?

Having thus found the general solution of the problem of the best critical region of the hypothesis H with regard to a chosen alternative h_1, we have now to consider whether this region is changed if instead of the hypothesis h_1 we take into consideration some other alternative simple hypothesis, say h_2. This question is very important for the following reasons. It has been shown by Neyman and Pearson that the tests based upon the best critical regions could not be bettered even if the *à priori* probability law of all unknown parameters were known. Therefore, in cases where for every admissible alternative hypothesis h the best critical region is the same, the problem of testing the hypothesis H may be considered as entirely solved: no better test can be found. If, however, the best critical regions with regard to different alternative hypotheses are different, then there is no "best" test of the hypothesis H. Whatever be the test, it could be bettered if the *à priori* probability law were known. There are intuitive reasons for believing that in these cases a "good" test is provided by the principle of likelihood.

The best critical region with regard to the simple alternative hypothesis h_1 being defined by the inequality (164), we have to consider if and under what conditions this inequality is independent of h_1.

It will be perhaps useful to recall the significance of all the symbols in (164):

$$nS_a^2 + nS_b^2 \quad \text{.....(165)}$$

denotes the minimum of the sum of squares

$$\sum_{i=1}^{n} (x_i - m_i)^2 = \sum_{i=1}^{n} (x_i - \sum_{j=1}^{s} c_{ji}a_j)^2 \quad \text{.....(166),}$$

calculated under the assumptions (1) that the x's are constant, (2) that the parameters

$$a_j = a_j^0 \qquad (j = 1, 2, \dots r) \quad \text{.....(167)}$$

have the values specified by the hypothesis tested, and (3) that the remaining parameters a_j for $j = r+1, r+2, \ldots s$ are arbitrary. It is seen that (165) does not depend upon the alternative hypothesis. Symbols R_{ij} for $i, j = 1, 2, \ldots r$ are also independent of the hypothesis h_1. According to formula (116),

$$nS_b^2 = \sum_{ij=1}^{r} R_{ij} E_i E_j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (168),$$

and the R's are the coefficients in the expression of nS_b^2 in terms of the E's and may be calculated in terms of the given coefficients c_{ij}. The meaning of the E's is given by the formula (55), i.e.

$$E_k = a_k - a_k^0 \qquad (k = 1, 2, \ldots r) \quad \ldots\ldots\ldots (169),$$

where a_k is a linear function of the x's giving the minimum value of the sum (166) if substituted for a_k, for $k = 1, 2, \ldots s$. Obviously E_k for $k = 1, 2, \ldots s$ does not depend upon the alternative hypothesis h_1. The only symbols depending upon h_1 in (164) are the ϵ's. In fact according to (113),

$$\epsilon_1 = a_i' - a_i^0 \qquad (i = 1, 2, \ldots r) \quad \ldots\ldots\ldots (170),$$

where a_i^0 is the value of the parameter a_i specified by H, and a_i' that specified by the alternative hypothesis h_1.

We see therefore that the question whether the best critical region may be independent of h_1 reduces to the question whether by some transformation the ratio

$$\frac{\sum_{ij=1}^{r} R_{ij} \epsilon_i E_j}{\sqrt{\sum_{ij=1}^{r} R_{ij} \epsilon_i \epsilon_j}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (171)$$

may be made independent of the ϵ's. This appears to be possible only if $r = 1$. In fact the numerator of (171) being a linear function of the ϵ's, we may hope to be able to cancel them only if the denominator were a rational polynomial in ϵ's. Now comparing the expression under the square root in (171) with (168), we see that it is a special value of the positive definite quadratic form nS_b^2 corresponding to

$$E_i = \epsilon_i \qquad i = 1, 2, \ldots r \quad \ldots\ldots\ldots\ldots\ldots (172),$$

and thus may be represented by a sum of at least r squares. Therefore the denominator in (171) is rational only if $r = 1$. We have thus the following interesting general result:

If the hypothesis H considered in this paper is either,

(1) *specifying the values of more than one of the parameters a_i $(i = 1, 2, \ldots s)$, or*

(2) *specifying the values of more than one linear function of the same parameters, then there is no common best critical region with regard to all alternatives h_1.*

It will be seen that there will be no common best critical region for instance when we test the following hypotheses:

(1) that the means of $k > 2$ sampled populations are equal;

(2) that the regression coefficients in $k > 2$ sampled populations are equal;

(3) that the regression lines in two (or more) sampled populations are identical. In fact, even if we compare only two regression lines, the hypothesis that they are identical can only be expressed by $r = 2$ equations.

In all these cases the best test available seems to be that provided by the principle of likelihood, discussed in § 4.

Let us now turn to the case $r = 1$ and consider whether there is a common best critical region with regard to all alternative hypotheses. In this case the ratio (171) reduces to

$$\frac{R_{11}\epsilon_1 E_1}{\sqrt{R_{11}}\,|\epsilon_1|} \quad \text{.....(173),}$$

and the inequality defining the best critical region (164), to

$$\frac{\sqrt{R_{11}}\,E_1}{\sqrt{nS_a^2 + R_{11}E_1^2}}\,\frac{\epsilon_1}{|\epsilon_1|} \geqslant v_0 \quad \text{or} \quad \frac{\sqrt{R_{11}}\,E_1}{\sqrt{n}\,S_a}\,\frac{\epsilon_1}{|\epsilon_1|} \leqslant \frac{v_0}{\sqrt{1 - v_0^2}} \quad \text{.....(174).}$$

It will be seen that in the most general case there will be no common best critical region even in this case. In fact, if

$$\epsilon_1 = a_1' - a_1^0 \quad \text{.....(175)}$$

may be for some admissible hypotheses positive, and for some others negative, then in the first case the best critical region will be defined by the inequality

$$\frac{\sqrt{R_{11}}\,E_1}{\sqrt{nS_a^2 + R_{11}E_1^2}} \geqslant v_0 \quad \text{.....(176),}$$

and in the second case by the inequality

$$\frac{\sqrt{R_{11}}\,E_1}{\sqrt{nS_a^2 + R_{11}E_1^2}} \leqslant -v_0 \quad \text{.....(177).}$$

In many cases, however, we may assume that the class of alternative hypotheses consists only of such hypotheses which ascribe a definite sign to the expression (175). In such cases there is a common best critical region, determined either by (176) or by (177).

As the left-hand side of the inequalities (173) and (174) may be written in the form

$$\frac{\sqrt{n}\,S_b}{\sqrt{nS_a^2 + nS_b^2}} \quad \text{.....(178),}$$

we see that the test may be based on the value of the ratio

$$Z = \frac{S_b}{S_a} \quad \text{.....(179),}$$

which, as was mentioned in § 4, follows "Student's" probability law. In this case the best critical region is determined by the inequality

$$Z\,\frac{\epsilon_1}{|\epsilon_1|} \geqslant z_0 \quad \text{.....(180),}$$

where z_0 should be found from the table of "Student's" integral to satisfy the condition

$$\int_{z_0}^{\infty}(1+z^2)^{-\frac{n-s+1}{2}}dz \Big/ \int_{-\infty}^{+\infty}(1+z^2)^{-\frac{n-s+1}{2}}dz = \epsilon \quad \text{.....(181).}$$

We may combine this with the results of Markoff, who proved that the expression, say,

$$\sigma_{a_1}^2 = \frac{nS_a^2}{n-s}\frac{1}{R_{11}} \quad \text{.....(182),}$$

is the estimate of the standard error squared of the variate a_1. Taking into account (169), we may now define the best critical region by means of the inequality

$$(a_1 - a_1^0)\frac{\epsilon_1}{|\epsilon_1|} \geq z_0\sqrt{n-s}\,\sigma_{a_1} \quad \text{.....(183).}$$

The above result may be used in two ways. Sometimes we may know the expression for σ_{a_1}. Then the best critical region may be found directly from (183). In other cases we do not know the value of σ_{a_1} and it may be desirable to calculate it for some purpose. Then we may do so by calculating nS_a^2 or the minimum of the sum of squares

$$\Sigma(x_i - m_i)^2 \quad \text{.....(184),}$$

considering all s parameters a as variable. Next we minimise the same sum with regard only to $s-1$ parameters, whilst one of them, $a_1 = a_1^0$, is kept constant. The result is $n(S_a^2 + S_b^2)$. Substituting, we obtain

$$nS_b^2 = R_{11}(a_1 - a_1^0)^2 \quad \text{.....(185).}$$

As a_1 must have been previously found in the process of getting nS_a^2, we may now calculate R_{11} and substitute it into (182) which will give us $\sigma_{a_1}^2$. This process will be illustrated below.

8. FURTHER EXAMPLES.

In this section I will give two further examples of linear hypotheses. In the first of them there exists a common best critical region with regard to all alternative hypotheses and in the other there does not.

Example 1. Denote by x and y the possible yields per acre given by some two varieties of cereals A and B. These yields depend largely upon the conditions of growth. If the two varieties are sown under similar conditions on adjacent plots, then both of them are similarly influenced and the yields thus obtained must be correlated. I shall suppose that experiments carried out in different experimental stations and possibly in different years gave the results

$$x_1, x_2, \ldots x_n, \quad \text{Mean} = \bar{x}, \quad \text{S.D.} = \sigma_x \quad \text{.....(186),}$$

$$y_1, y_2, \ldots y_n, \quad \text{Mean} = \bar{y}, \quad \text{S.D.} = \sigma_y \quad \text{.....(187).}$$

I shall further assume as known that the regression of y on x is linear*. The

* This has been found in many cases studied by M. Gorski and K. Iwaszkiewicz. See *Roczniki Nauk Rolniczych i Leśnych*, t. XXVIII. pp. 211 *et seq.* and t. XXXI. pp. 277 *et seq.* See also W. Staniszkis, "Ergebnisse dreijähriger (1927—29) vergleichender Versuche mit Leinsorten," *Wyniki Doswiadczen Polowych*, No. 6, Warsaw, 1934.

hypothesis, H, to be tested consists in the assumption that in conditions where the variety A is able to give a specified yield of X tons per acre, the average yield of the variety B does not exceed the value Y, which is also specified. Usually it will probably be assumed that $Y = X$.

If after having applied the test we reject the hypothesis H, this will be equivalent to the assumption that the experimental data give us sufficient evidence that in conditions when the yield of A is X, the variety B is better.

Let us now present the hypothesis H under the form of a linear hypothesis. We start by noticing that the assumption about the linearity of regression of y on x means that if we fix any value of x and consider the population, say π_x, of all possible y's, then the mean of this population, say m_x, will be connected with the fixed value of x by means of the formula

$$m_x = a_1 + a_2 x \quad \text{.....(188),}$$

where a_1 and a_2 are independent of x. This equation, which must hold for any value of x, and therefore for any value of this variable given in the sample, is equivalent to my equation (4).

The rôle of the variates (1) is now played by the yields of the variety B, denoted by y. The hypothesis to test is expressed by the equation

$$m_X = a_1 + a_2 X = Y \quad \text{.....(189),}$$

while the alternative hypotheses are

$$m_X' = a_1 + a_2 X > Y \quad \text{.....(190).}$$

The equation (189) corresponds to the equation (6) of the theory. If there is evidence that the standard deviation in each array of y is independent of x, then it is easily seen that the hypothesis considered here is a linear hypothesis. The number of unknown parameters $s = 2$. These are the coefficients a_1 and a_2 of the regression equation (188). The hypothesis itself is expressed by means of only one equation (189). Therefore $r = 1$. As the alternatives satisfy the inequality (190), we conclude that there is a common best critical region with regard to all alternatives. It may be defined by the inequality (183). In our case a_1^0 means the hypothetical value Y of the ordinate m_X of the regression line. a_1 is its value as obtained by method of least squares,

$$a_1 = \bar{y} + r\frac{\sigma_y}{\sigma_x}(X - \bar{x}) \quad \text{.....(191),}$$

where r means the correlation coefficient between (186) and (187). ϵ_1 will be the difference between the value of the ordinate m_X as specified by the alternative hypotheses and that specified by the hypothesis tested. Thus $|\epsilon_1| = \epsilon_1 > 0$. The value of σ_{a_1} may be calculated by the formula of Miss K. Iwaszkiewicz (see Neyman (7)) or again it may be directly calculated following the method described in the last paragraphs of the preceding section. For the sake of illustration we shall follow this latter method.

We consider the sum of squares

$$n\chi^2 = \Sigma (y_i - a_1 - a_2 x_i)^2 \quad \text{.....(192)}$$

and find its minimum with regard to a_1 and a_2. This is known to be

$$nS_a^2 = n\sigma_y^2 (1 - r^2) \quad \text{.....(193).}$$

Next we assume that a_1 and a_2 fulfil the equality (189), substitute in (192)

$$a_1 = Y - a_2 X \quad \text{.....(194),}$$

and minimise, say, $n\chi'^2 = \Sigma \{y_i - Y - a_2 (x_i - X)\}^2$ .....(195)

with regard to a_2. Easy algebra gives the result:

$$n(S_a^2 + S_b^2) = n\left\{\sigma_y^2 (1 - r^2) + \frac{\sigma_x^2}{\sigma_x^2 + (X - \bar{x})^2}\left(\bar{y} + r\frac{\sigma_y}{\sigma_x}(X - \bar{x}) - Y\right)^2\right\}$$

.....(196).

Comparing (196) with (193), (185), and (191), we find

$$R_{11} = \frac{n\sigma_x^2}{\sigma_x^2 + (X - \bar{x})^2} \quad \text{.....(197).}$$

Now (182) gives $\sigma_{a_1}^2 = \frac{\sigma_y^2 (1 - r^2)}{n - 2}\left(1 + \frac{(X - \bar{x})^2}{\sigma_x^2}\right)$ .....(198)

and finally the best critical region is determined by the inequality

$$\bar{y} + r\frac{\sigma_y}{\sigma_x}(X - \bar{x}) - Y \geqslant z_0 \sqrt{n - 2}\, \sigma_{a_1} \quad \text{.....(199),}$$

where z_0 satisfies the condition (181).

Example 2. *The hypothesis about the linearity of regression.* Suppose that the observations gave us n pairs of values of two variables x and y. These pairs are grouped according to different values of x, which are

$$x_1, x_2, \ldots x_s \quad \text{.....(200).}$$

The values of y which correspond to the common value of x, say x_i, will be denoted by

$$y_{i1}, y_{i2}, \ldots y_{in_i}, \text{ Mean} = \bar{y}_i, \sum_{i=1}^{s} n_i = n \quad \text{.....(201).}$$

I shall suppose that it is known that the standard deviation of the population of all y's corresponding to a fixed value of x is independent of the value of x and that the distribution of y's in the arrays is normal. The hypothesis, H, to test is that the regression of y on x is linear.

We shall start by showing that H is what I call a linear hypothesis. Denote by m_x the population mean of y's corresponding to a fixed value of x. That is to say, m_x will be the mean of an array of y's in the population sampled. The hypothesis H to be tested is expressed by the equation

$$m_x = A + Bx \quad \text{.....(202),}$$

where A and B are some constant numbers and x has any value. This equation may be considered as corresponding to the equations (6) in the theory. In order

to write the equations corresponding to (4) we shall denote by m_{ij} the population mean of the variate y_{ij}. As this corresponds to the value x_i of x, we shall sum up our knowledge concerning m_{ij} by writing

$$m_{ij} = m_i \quad \text{.....(203)}$$

for any $j = 1, 2, \ldots n_i$ and $i = 1, 2, \ldots s$. Thus we see that the population means of all variables y are linearly expressed in terms of s unknown parameters m_i. Here we see that a test of the hypothesis H is possible only if there is at least one array in which we have at least two observations. In fact, if for any i we had $n_i = 1$, then we should have $s = n$, contrary to the assumption of the theory that the number of parameters is less than the number of observations.

Now, as the rôle of the unknown parameters is played by the m_i's, we shall have to transform the equation (202) in order to bring it into correspondence with the equations (6), which are linear with regard to parameters and then contain only known numbers b and B. Substituting in (202) $x = x_1$, $x = x_2$ and $x = x_i$ for $i = 3, 4, \ldots s$, we get

$$\left.\begin{aligned} m_i - A - Bx_i &= 0 \\ m_1 - A - Bx_1 &= 0 \\ m_2 - A - Bx_2 &= 0 \end{aligned}\right\} \quad \text{.....(204),}$$

a system of linear equation which allows the elimination of A and B. The result is given by

$$\begin{vmatrix} m_i, & 1, & x_i \\ m_1, & 1, & x_1 \\ m_2, & 1, & x_2 \end{vmatrix} = 0, \quad \text{for } i = 3, 4, \ldots, s \ \ldots \ (205),$$

or
$$m_i(x_2 - x_1) + m_1(x_i - x_2) + m_2(x_1 - x_i) = 0, \quad \text{for } i = 3, 4, \ldots s \quad \ldots(206).$$

Now we see another condition which the observations must satisfy to allow the test of the hypothesis H. The number of the arrays, or s, must be at least equal to 3. Otherwise the equations (206), which express the hypothesis H, could not have been written. If $s \geqslant 3$, then the number, r, of the equations expressing the hypothesis H is $r = s - 2$.

It is easily seen that in the case considered there is no common best critical region with regard to all alternatives. In fact, even if $s = 3$ and thus $r = 1$, the class of alternative hypotheses would certainly include those which ascribe a value of m_3 such that

$$m_3(x_3 - x_1) > -m_1(x_i - x_2) - m_2(x_1 - x_i) \quad \text{.....(207),}$$

and also those for which

$$m_3(x_3 - x_1) < -m_1(x_i - x_2) - m_2(x_1 - x_i) \quad \text{.....(208).}$$

Therefore the test to be applied is that following from the principle of likelihood, defined by (72) or by

$$\frac{S_b^2}{S_a^2} \geqslant \frac{1}{u_0} - 1 \quad \text{.....(209),}$$

where u_0 must be found from the Tables of the Incomplete Beta Function so as to satisfy the condition

$$B_{u_0}\left(\frac{n-s}{2}, \frac{r}{2}\right) = \epsilon B\left(\frac{n-s}{2}, \frac{r}{2}\right) \quad \text{.....(210).}$$

Now S_a^2 and S_b^2 are found as follows. We consider the sum of squares

$$n\chi^2 = \sum_{i=1}^{s} \sum_{j=1}^{n_i} (y_{ij} - m_i)^2 \quad \text{.....(211),}$$

and minimise it with regard to all m_i for $i = 1, 2, \ldots s$. This gives us

$$nS_a^2 = \sum_{i=1}^{s} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 \quad \text{.....(212).}$$

Next we assume that the hypothesis tested is true and minimise (211) with regard to what is not specified by the hypothesis. For this purpose we could use the equations (206) and consider m_1 and m_2 as being not specified by H. But obviously we may use instead the expression

$$m_i = A + Bx_i \quad \text{for} \quad i = 1, 2, \ldots s \quad \text{.....(213),}$$

and consider A and B as numbers which are not specified by H. The minimum of (211) calculated under the conditions (213) is then, as it is well known,

$$n(S_a^2 + S_b^2) = n\sigma_y^2(1 - r^2) \quad \text{.....(214),}$$

where σ_y means the standard deviation of all y's in the sample, and r the correlation between x and y in the sample. Thus

$$nS_b^2 = n\sigma_y^2(1 - r^2) - \sum_{i=1}^{s} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 \quad \text{.....(215),}$$

and the critical region is determined by the inequality, say,

$$u = \frac{\sum_{i=1}^{s} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2}{n\sigma_y^2(1 - r^2)} \leqslant u_0 \quad \text{.....(216).}$$

9. SUMMARY OF RESULTS.

1. The problem considered in the present paper consists in testing a composite hypothesis H, which may be described as a "linear hypothesis concerning the means of n normal variates with common standard deviation." It is supposed that the mean, m_i, of each of the n normal variates, x_i, having the same unknown standard deviation σ, are linear functions with known coefficients of s parameters a_i $(i = 1, 2, \ldots s)$. The values of parameters are not known and the hypothesis H to be tested consists in the assumption, either (1) that some $r \leqslant s$ of these parameters have some specified values, say $a_1^0, a_2^0, \ldots a_r^0$, the remaining parameters being unspecified, or (2) that some $r \leqslant s$ independent linear functions, θ_i, of these parameters with known coefficients have some definite values, say $\theta_1^0, \theta_2^0, \ldots \theta_r^0$.

2. It was possible to give a general solution of the above problem, based upon the principle of likelihood, proposed by J. Neyman and E. S. Pearson.

3. A general expression has been given for the best critical region for the hypothesis H with regard to any simple alternative hypothesis h_1.

4. It has been shown that in cases where the number r of the parameters a_i (or of their linear functions) which are specified by the hypothesis H exceeds one, then there is no best critical region common to all possible alternative hypotheses. In those cases therefore the method of testing the hypothesis H must be based upon the principle of likelihood—unless some other "good" test is invented.

5. If $r = 1$ and if the class of admissible hypotheses is limited to those ascribing to the parameter, say a_1, or to the linear function, say θ_1 (which are specified by H) values which are larger (or smaller) than the values specified by H, then there exists a best critical region, common to the whole class of the alternatives. In these cases the solution obtained of the problem of testing H could not be bettered even if the probability law *à priori* of all unknown parameters were known.

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EIN BEITRAG ZU UNTERSUCHUNGEN UEBER ZWEIDIMENSIONALE VERTEILUNGEN VON MASSENPUNKTEN BEI ZUFALLSARTIG BEDINGTEN BEWEGUNGEN*.

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Einleitung.

In dem Grundproblem der vorliegenden Arbeit handelt es sich um die Bestimmung einer Verteilungsdichte, wenn von einem Flächenelement mit dem inneren Punkt Q—dem Quellpunkt—ausgehend N Massenpunkte sich über eine Ebene in folgender Weise verteilen: Durch Q mögen m verschiedene Geraden $G_1, G_2, G_3, \ldots, G_m$ gehen, die m Bewegungsrichtungen repräsentieren. Auf jeder dieser Geraden gelte ein Wahrscheinlichkeitsgesetz $v_r(s_r)$, das ist die Wahrscheinlichkeitsdichte für die Länge einer Bewegung auf G_r oder parallel dazu. $v_r(s_r)\,ds_r$ ist also die Wahrscheinlichkeit, um auf einer Parallelen zu G_r von einem als Anfangspunkt genommenen Punkt P_{r-1} bis zu einem Punkt P_r zu kommen, der (von P_{r-1} ausgemessen) zwischen s_r und $s_r + ds_r$ liegt, dabei ist s_r irgendein Parameter, der eineindeutig den Punkten der Geraden G_r zugeordnet ist. Ein Massenpunkt M_i bewege sich nun zunächst von Q ausgehend längs G_1 gemäss der dortigen Verteilungsfunktion $v_1(s_1)$ bis zu einem P_1, darauf von P_1 ausgehend parallel G_2 bis zu einem Punkt P_2 gemäss $v_2(s_2)$ usw. Allgemein bewege sich M_i von P_{r-1} ausgehend parallel G_r bis zu einem Punkt P_r gemäss der auf G_r geltenden Wahrscheinlichkeitsdichte $v_r(s_r)$. Nach m solchen Schritten, deren Reihefolge beliebig ist, da es sich um einfache Vektoraddition handelt, befindet sich M_i an einem Punkt $P_m = P$. Der analoge Vorgang finde für alle N Massenpunkte statt. Gefragt ist dann nach der Dichte $\Phi_1(P)$, d.h. nach der Anzahl der Massenpunkte pro Flächeneinheit an einem Punkt P der Ebene und zwar bei grossem m und N.

Es sind dabei alle m Richtungen als gleich wahrscheinlich angenommen; ist das jedoch nicht der Fall, so lässt sich dieser auf den obigen zurückführen, indem man die $v_r(s_r)$ entsprechend ändert. Sei z.B. im extremen Fall die Bewegung parallel einer bestimmten Geraden G_r unmöglich, so kann man das ausdrücken, indem man setzt:

$$\lim_{ds_r \to 0} v_r(s_r)\,ds_r = 1 \text{ für den Punkt } Q \text{ (oder den entsprechenden Punkt } P_{r-1}),$$

$$v_r(s_r) = 0 \text{ für alle anderen Punkte.}$$

Aus der Verteilung $\Phi_1(P)$ ergibt sich nun eine weitere, indem von *allen* Flächenelementen df mit einem inneren Punkt $\bar{Q}$ $\Phi_1(\bar{Q})\,df$ Massenpunkte in

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derselben Weise ausgehen wie vorher die N Massenpunkte von Q aus. Die Verteilungsdichte der $\Phi_1(\bar{Q})\,df$ Massenpunkte um $\bar{Q}$ ist also

$$\Phi_1(\bar{Q})\,df\,\Phi_1(\bar{Q}P):N,$$

dabei ist $\Phi_1(\bar{Q}P)$ gleich $\Phi_1(P)$ bezogen auf $\bar{Q}$ statt Q als Anfangspunkt. Die Ueberlagerung aller dieser Verteilungen ergibt dann

$$\Phi_2(P)=\frac{1}{N}\int\Phi_1(\bar{Q}P)\,\Phi_1(Q)\,df.$$

Allgemein erhält man die n-te Verteilung durch die Rekursionsformel

$$\Phi_n(P)=\frac{1}{N}\int\Phi_1(\bar{Q}P)\,\Phi_{n-1}(\bar{Q})\,df,$$

wobei die Integrale beidemal über die ganze Ebene zu erstrecken sind.

Nach Lösung dieses Grundproblems soll dann die Verteilung berechnet werden, wenn sich von allen Punkten einer begrenzten Fläche mit der Dichte N die Massenpunkte in der obigen Weise über die Ebene ausbreiten, wobei mit Rücksicht auf die Anwendungen einige Spezialfälle in bezug auf die Art der Begrenzung näher ausgeführt werden sollen.

Ueber die biologische Bedeutung der folgenden Ausführungen möchte ich an dieser Stelle noch einige Erläuterungen geben. Es handelt sich in biologischer Hinsicht bei der vorliegenden Arbeit im wesentlichen um folgende zwei hauptsächlich interessierende Fragen:

1. Wie viele Moskitos befinden sich durchschnittlich pro Flächeneinheit in verschiedenen Entfernungen von einem stark Moskito-verseuchten Gebiet, wenn nur in diesem eine Vermehrungsmöglichkeit für jene vorhanden ist? Dabei möge die Ausbreitung der Insekten nur durch zufälliges Hin- und Herfliegen zustande kommen. Von irgend welchen grösseren bewussten oder absichtlichen "Zielflügen" und von äusseren Hemmnissen sei also zunächst abgesehen. Hier angenommene Verhältnisse liegen z.B. annähernd in dem Steppengelände des Wolgagebietes vor. Dort grenzen an die Sumpfniederungen der Wolga, die stark mit Moskitos bevölkert sind, verhältnismässig grosse Steppengebiete, die den Moskitos eigentlich nur in der Nähe menschlicher Siedlungen, durch die dort vorhandenen Tiertränken oder Wasserlachen, eine Brutmöglichkeit bieten. Werden diese nun desinfiziert (z.B. durch Bestreuen mit Arsenstaub), so kann eine Malaria-Uebertragung nur durch die aus den Wolganiederungen eingestreuten Moskitos zustande kommen. Für die Bekämpfung der Malaria ist es in einem solchen Fall daher ausserordentlich nützlich zu wissen, wie weit eine beträchtliche Anzahl dieser Moskitos in das Steppengelände eindringt. Ein beliebig weites Eindringen ist des allmählichen Absterbens wegen nicht möglich, da ausserhalb des Flussgürtels wohl Gelegenheit zur Eiablage, nicht aber, oder in verschwindend geringem Masse, Möglichkeiten zur Vollendung der Entwicklung vorhanden sind. Auf die Wichtigkeit dieser Frage für die Malariabekämpfung wurde bereits von Sir Ronald Ross hingewiesen, der auch schon eine Verteilungsfunktion zu berechnen versuchte, worauf später noch zurückzukommen sein wird.

In diesem Zusammenhang möchte ich darauf hinweisen, dass die Ausführungen dieser Arbeit sich nicht auf solche Tierarten beziehen, die bei ihren "Flügen" infolge irgendeines Instinkts absichtlich—wie z.B. die Bienen bei ihren täglichen Flügen zur Honigsuche—an den Ausgangspunkt zurückkehren, sondern ein Zurückkommen an den Ursprung sei reiner Zufall. In obigem Fall ist z.B. wegen der Möglichkeit der Eiablage in der Steppe ein regelmässiges Zurückfliegen der Moskitos an die Brutplätze des Flussgebietes selbst dann nicht wahrscheinlich wenn für jene ein sogenannter "Heimatsinn" anzunehmen wäre, wobei als Heimat im Fall der Moskitos Gegenden mit Brutmöglichkeiten anzusehen wären. Die Möglichkeit des Vorhandenseins eines "Heimatsinns" ist im übrigen bei den meisten Moskitoarten noch durchaus ungeklärt.

2. Wie wird sich die Ausbreitung (Infiltration) einer Tierart in ein zunächst unbesiedeltes Gebiet, das an ein besiedeltes grenzt, abspielen? Dabei ist wieder die wichtigste Voraussetzung, dass die Ausbreitung nur durch zufälliges Hin- und Herfliegen erfolgt. Weiter sei angenommen, dass *alle* Punkte der Ebene als "Plätze mit Vermehrungsmöglichkeiten" oder als "habitats" (in dem Sinne der noch zu besprechenden Arbeit von Herrn Prof. K. Pearson) möglich sind oder zumindest, dass die Entfernungen der einzelnen "habitats" sehr klein sind gegenüber der durchschnittlichen Länge eines Fluges. Dieses Problem ist natürlich nur als ein dynamisches zu betrachten, denn der stationäre Endzustand wird in dem hier angenommenen einfachsten Fall der ortsunabhängigen Vermehrungsmöglichkeit, örtlich konstant sein. Es kann sich also nur darum handeln, den zeitlichen Verlauf einer oben beschriebenen Besiedlung eines zunächst leeren Gebietes darzustellen, und zwar, wie gesagt, eines Gebietes, das an fast allen Punkten Siedlungsmöglichkeiten bietet. Auf den Fall, dass solche Siedlungsmöglichkeiten nur an vereinzelten Stellen, deren Entfernungen in die Grössenordnung einer Fluglänge oder mehr kommen, vorhanden sind, sind also die folgenden Ergebnisse nicht oder höchstens als erste grobe Annäherung anwendbar.

1. *Voraussetzungen und Ergebnisse der älteren Arbeiten.* (*Ronald Ross, Lord Rayleigh, Karl Pearson.*)

In diesen Arbeiten handelt es sich um ein ähnliches wie das in der Einleitung diskutierte Problem. Die wesentlichen biologischen Voraussetzungen, wie sie in der Arbeit von Herrn Professor Pearson angegeben wurden (s. Lit.-Verz. Nr. (3), S. 3 und 4), seien im folgenden kurz dargestellt.

Ueber eine genügend grosse Fläche seien gleichförmig sogen. "habitats"—d.h. für Mücken: Plätze mit Brut- und Lebensmöglichkeiten—verteilt. Die durchschnittliche Fluglänge l einer Mückenart von einem "habitat" zu einem neuen möge als "flight" bezeichnet werden. Diese "flights" sind zu unterscheiden von den "flitters," die ein blosses Hin- und Herfliegen in der Nachbarschaft der "habitats" zwecks Nahrungssuche etc. bedeuten.

Die Frage des Grundproblems lautet dann: Wie gross ist die Anzahl der Mücken pro Flächeneinheit—also die Dichte $\Phi_n(P)$—an einem Punkt P in der

Entfernung r von Q, wenn von Q ausgehend N Mücken n "flights" der Länge l (= durchschnittlicher Abstand benachbarter "habitats") ausführen? Dabei soll es sich um sogen. "Random Migration" handeln, d.h. die Verteilung soll durch zufälliges Hin- und Herfliegen entstehen. Die einzelnen Flugstrecken (von einem "habitat" zum anderen) seien jedoch gradlinig zurückgelegt und wie beim ersten Flug seien von einem Ausgangspunkt aus alle Richtungen gleich wahrscheinlich. Beim ersten Fluge verteilen sich also N Mücken in erster Näherung gleichmässig um Q auf einen schmalen Kreisring mit dem Radius l. Beim zweiten Flug wiederholt sich das von jedem Flächenelement des ersten Kreisringes, und die Verteilungen auf den nun entstehenden Kreisringen überlagern sich und ergeben die Verteilung nach dem zweiten Flug. In analoger Weise erhält man alle weiteren Verteilungen.

Sir Ronald Ross führt die Frage auf ein lineares Problem zurück. Er macht anscheinend die Annahme (das ist in seiner Arbeit jedoch nicht genau angegeben), dass nach einem Fluge von Q aus—bei Projektion der einzelnen Flüge auf die X-Achse—$\frac{1}{2}N$ der Mücken in Q geblieben sind, während je $\frac{1}{4}N$ sich im Abstand l links bzw. rechts von Q befinden. Von diesen drei Punkten wiederholt sich jetzt dieser Vorgang, nur dass statt N jetzt $\frac{1}{2}N$ bzw. $\frac{1}{2}N$ Mücken sich verteilen, so dass nach dem zweiten Fluge in Q: $\frac{6}{16}N$ und rechts und links von Q im Abstand $1l$ je $\frac{4}{16}N$ und im Abstand $2l$ je $\frac{1}{16}N$ Mücken sich befinden. Das gibt dann nach n Flügen die bekannte Verteilung der Binomialkoeffizienten von 2^{2n} multipliziert mit $N(\frac{1}{2})^{2n}$. Es befinden sich also nach n Flügen im Abstand kl von Q insgesammt

$$\left.\begin{aligned} 2\binom{2n}{n-k}N(\tfrac{1}{2})^{2n} \quad & \text{für } k>0 \\ \binom{2n}{n-k}N(\tfrac{1}{2})^{2n} \quad & \text{für } k=0 \end{aligned}\right\} \quad \text{.....(I).}$$

Wie aus der Wahrscheinlichkeitsrechnung bekannt gilt nun für grosse n und $n \gg k$

$$\binom{2n}{n-k}\frac{2N}{2^{2n}} \sim \frac{2N}{\sqrt{n\pi}}e^{-\frac{k^2}{n}} \quad \text{.....(II).}$$

Das muss also, da in Wirklichkeit eine kontinuierliche Verteilung besteht, die Anzahl Mücken sein, die sich nach dem n-ten Fluge auf einem Kreisring der Breite l, der den Kreis mit dem Radius kl umgibt, befinden. Bedeutet also $\Phi_n(r)$ die Dichte im Abstand r von Q, so muss sein:

$$\begin{aligned} \frac{2N}{\sqrt{n\pi}}e^{-\frac{k^2}{n}} &= \int_0^{2\pi}\int_{(k+\delta-1)l}^{(k+\delta)l}\Phi_n(r)\,r\,dr\,d\phi \qquad \text{wobei } 0 \leqslant \delta \leqslant 1 \\ &= 2\pi\int_{(k+\delta-1)l}^{(k+\delta)l}\Phi_n(r)\,r\,dr \\ &= 2\pi\Phi_n([k+\epsilon]\,l)\int_{(k+\delta-1)l}^{(k+\delta)l} r\,dr \qquad \text{wobei } \delta-1 \leqslant \epsilon \leqslant \delta \\ &= 2\pi l^2(k+\delta-\tfrac{1}{2})\,\Phi_n([k+\epsilon]\,l) \quad \text{.....(III),} \end{aligned}$$

also muss die Dichte für ein r, dass zwischen $(k-1)\,l$ und $(k+1)\,l$ liegt, sein

$$\Phi_n([k+\epsilon]\,l) = \frac{Ne^{-\frac{k^2}{n}}}{l^2(k+\delta-\frac{1}{2})\,\pi\sqrt{\pi n}} \quad \text{.....(IV).}$$

Diese Reduktion des zweidimensionalen Problems auf ein eindimensionales ist jedoch wohl reichlich willkürlich, und wieso dieses Resultat mit den späteren von Lord Rayleigh (V) und K. Pearson (VI) für grosse n übereinstimmt, wie Ronald Ross behauptet, ist wohl kaum einzusehen, denn jene erhalten für grosse n:

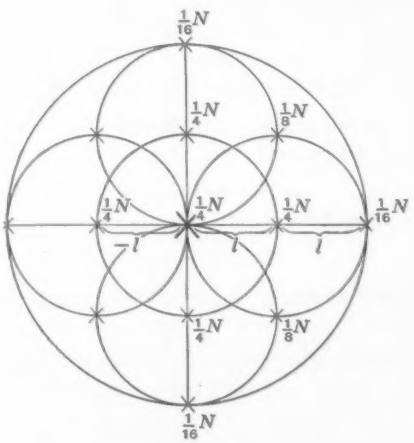
$$\Phi_n(r) = \frac{N}{nl^2\pi}\,e^{-\frac{r^2}{nl^2}},$$

also

$$\Phi_n([k+\epsilon]\,l) = \frac{N}{nl^2\pi}\,e^{-\frac{(k+\epsilon)^2 l^2}{nl^2}}.$$

Da nun sowohl δ wie ϵ nur zwischen -1 und $+1$ variieren können, geht dieser letzte Ausdruck mit $n \to \infty$ etwa wie $1:n$ gegen 0, während (IV) nur wie $1:\sqrt{n}$ gegen 0 geht. Der absolute Unterschied zwischen diesen beiden Resultaten ist allerdings für grosse n nur klein, da ja beide gegen 0 gehen; relativ unterscheiden sie sich jedoch der Grössenordnung nach etwa um den Faktor $\frac{1}{k}\sqrt{\frac{n}{\pi}}$, der mit wachsendem n also beliebig gross werden kann, und nur bei $k \sim \sqrt{\frac{n}{\pi}}$ ist eine ziemliche Uebereinstimmung zu erwarten.

Etwas näher kommt der gestellten Aufgabe das von K. Pearson zitierte Resultat einiger Arbeiten von Lord Rayleigh über akustische Probleme (s. Lit.-Verz. Nr. (2)). Hierbei sind *zwei* Richtungen (x- und y-Achse) für den von einem Punkt ausgehenden gradlinigen Flug l als gleich wahrscheinlich angenommen. Ebenfalls sollen die Wahrscheinlichkeiten für einen Flug nach der positiven und negativen Seite der x- bzw. y-Achse hin gleich sein. Für grosse N sind also folgende Verteilungen zu erwarten:



Nach dem *ersten* Flug befinden sich in den 4 Punkten (×), siehe S. 195:

$$x=0,\quad y=\pm l;\quad x=\pm l,\quad y=0,$$

je $N:4$ und in allen andern Punkten der Ebene $0\times N$ Mücken.

Von diesen vier Punkten verteilen sich nun die $N:4$ wie vorher die N Mücken von Q aus, so dass nach dem *zweiten* Fluge sich in den Punkten (×):

$$\begin{aligned} &x=0,\quad y=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad N:4,\\ &x=0,\quad y=\pm 2l;\quad x=\pm 2l,\quad y=0 \quad \ldots \quad \text{je } N:16,\\ &x=\pm l,\quad y=\pm l \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{je } N:8 \end{aligned}$$

Mücken befinden, wie man leicht aus obiger Skizze entnimmt. Analog erhält man alle weiteren Verteilungen. Wenn $n\to\infty$ geht und gleichzeitig l gegen 0, so dass nl^2 endlich bleibt, ergibt sich eine Verteilung mit der Dichte

$$\Phi_n(r)=\frac{N}{\pi n l^2}\,e^{-\frac{r^2}{nl^2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(V)},$$

die jedoch nur für grosse n und $r^2 \ll nl^2$ angenähert gilt.

Die exakte Lösung dieses allgemeinen Problems—also die der Verteilung bei Annahme einer *konstanten* Fluglänge mit Gleichwahrscheinlichkeit für *alle* Richtungen—hat dann Professor Karl Pearson 1906 gegeben (s. Lit.-Verz. Nr. (3)). Vorher hatte I. C. Kluyver einen allgemeinen Ansatz mit Hilfe Besselscher Funktionen aufgestellt, der nicht zum Ziele führte, jedoch die Lösung einiger spezieller Probleme und den Beweis für die Richtigkeit der Rayleigh'schen Lösung für grosse n erbrachte (s. Lit.-Verz. Nr. (4)). K. Pearson erhielt als Lösung ebenfalls ein Integral über ein Produkt Besselscher Funktionen (näheres über Besselsche Funktionen s. Lit.-Verz. Nr. (5), (6)), nämlich:

$$\Phi_n(r)=\frac{N}{2\pi}\int_0^\infty u J_0(ur)\,\{J_0(ul)\}^n du \qquad \ldots\ldots\ldots\ldots\text{(VI)},$$

was sich durch geeignete Umformung in folgende Reihe entwickeln lässt:

$$\Phi_n(r)=N(\nu_0\omega_0+\nu_2\omega_2+\nu_4\omega_4+\ldots) \qquad \ldots\ldots\ldots\ldots\text{(VII)},$$

wobei

$$\omega_{2s}=-\frac{(-\beta)^{s+1}}{\pi n l^2}\,\frac{d^s}{d\beta^s}\left(\frac{1}{\beta}e^{\frac{1}{\beta}}\right) \qquad \ldots\ldots\ldots\ldots\text{(VIII)},$$

mit

$$\beta=-\frac{nl^2}{r^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(IX)},$$

und

$$\nu_{2s}=(-1)^s\,\frac{\pi n^{s+1} l^{2(s+1)}\int_0^\infty \omega_{2s}\Phi_n(r)\,e^{\frac{r^2}{nl^2}} r\,dr}{N\int_0^\infty \omega_{2s} r^{2s+1} dr} \qquad \ldots\ldots\ldots\ldots\text{(X)}.$$

Die bestimmten Integrale in diesem Ausdruck lassen sich auswerten, man erhält z.B. für

$$\nu_0=1,\quad \nu_2=0,\quad \nu_4=-\frac{1}{4n},\quad \nu_6=-\frac{1}{9n^2},$$

$$\nu_8=\frac{6n-11}{12n^3},\quad \nu_{10}=\frac{50n-57}{1800n^4}\ \ldots.$$

Die Konvergenz der Reihe der ω-Funktionen ist nun für $n < 7$ sehr schlecht. Für diesen Fall haben K. Pearson und J. Blakeman die Verteilungsfunktion auf graphischem Wege gefunden. Für grosse n geht

$$\Phi_n(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{V } a),$$

gegen $N\omega_0$, das ist die Rayleigh'sche Lösung.

Den bisher besprochenen Methoden lag, wie gesagt, die Annahme einer *konstanten* Fluglänge zugrunde. Vielfach wird in der Natur diese Annahme nicht berechtigt sein, sondern häufig werden sich schon nach der ersten Verteilung von einem Zentrum aus in *allen* Entfernungen die entsprechenden Lebewesen vorfinden, vorausgesetzt dass kein Punkt durch besonders günstige oder ungünstige Lebensbedingungen ausgezeichnet ist. In diesem Fall ist es nun durchaus nicht evident, dass nach n Flügen dieselbe Verteilung resultiert, wenn man nach einem Fluge von einem Zentrum aus annimmt: (1) Alle Mücken befinden sich auf einem schmalen Kreisring mit dem Radius l, oder (2) wenn mann annimmt, irgendeine Verteilung mit derselben *durchschnittlichen* Fluglänge l erstreckt sich über die ganze umliegende Fläche. Während beim ersten Fall die Verteilungen nur von Flächenelementen des Kreisringes ausgehen, die sich dann überlagern, findet dieser Vorgang in Wirklichkeit vielleicht von allen Flächenelementen einer ganzen Umgebung des Zentrums statt, und es ist nicht gesagt, dass die Verteilungen ausgehend von Flächenelementen ausserhalb bzw. innerhalb des Kreises mit dem Radius l sich derart superponieren, als ob sie alle von den Flächenelementen des Kreisringes mit dem Radius der durchschnittlichen Fluglänge l ausgingen.

2. *Die Grundverteilung nach dem ersten Fluge um ein gegebenes Zentrum bei* variabler *Fluglänge*.

In dieser Arbeit soll mit möglichst wenigen und plausiblen Voraussetzungen die Verteilungsfunktion bei variabler Fluglänge gefunden werden. Ich gehe dabei gemäss der in der Einleitung ausgeführten Problemstellung von folgenden Voraussetzungen aus, um die Verteilung nach dem ersten Fluge zu erhalten:

1. Es seien m verschiedene Richtungen (wobei nachdem $m \to \infty$) möglich, in die eine Mücke fliegt. Die Richtungen seien dargestellt durch m Geraden durch den Ursprung.

2. Die Fluglängen einer bestimmten Richtung G_r unterliegen einem Wahrscheinlichkeitsgesetz $v_r(s_r)$*.

3. Während des Fluges wechselt die Mücke m-mal ihre Richtung und durchfliegt in der jeweiligen Richtung eine gemäss $v_r(s_r)$ zu erwartende Strecke.

4. Die Wahrscheinlichkeitsdichten $v_r(s_r)$ auf den Geraden G_r sind ausserhalb eines Kreises mit dem Radius K um den Ursprung Null.

* $v_r(s_r)$ braucht im einzelne.. nicht bekannt zu sein.

5. Die $v_r(s_r)$ mögen für alle r einer Lipschitzbedingung genügen, d.h. die zu den einzelnen $v_r(s_r)$ gehörigen Zahlen M_r besitzen eine obere Schranke M; es gilt also für beliebige Punkte der Geraden G_r:

$$\left| \frac{v_r(s_{r1}) - v_r(s_{r2})}{s_{r1} - s_{r2}} \right| < M_r < M.$$

Die letzten beiden Voraussetzungen erwähne ich nur, da sie beim Beweis benutzt werden, während sie für die eigentliche Problemstellung unwesentlich sind und nur die Art der sonst beliebigen Funktionen $v_r(s_r)$ etwas einschränken.

Unser Problem ist dann identisch mit der Frage nach der Wahrscheinlichkeitsdichte $w_m(QP)$ eines Punktes P (des Endpunktes des von Q aus abgetragenen Vektors $\overrightarrow{QP}$), den ich erhalte, wenn ich bilde:

$$\overrightarrow{QP}_1 + \overrightarrow{QP}_2 + \overrightarrow{QP}_3 + \ldots + \overrightarrow{QP}_m = \overrightarrow{QP},$$

wobei die Punkte P auf den Geraden G_r gemäss $v_r(s_r)$ variieren.

Fast dasselbe Problem ist nun für den k-dimensionalen Raum von H. Pollaczek-Geiringer (s. Lit.-Verz. Nr. (7)) behandelt worden, nur variieren dort die Punkte P_r auf geschlossenen Kurven C_r statt wie bei uns auf den Geraden G_r. Für den Beweis ist davon aber nur von Bedeutung, dass man durch die Annahme geschlossener Kurven in einem endlichen Gebiet bleibt, was in unserm Fall durch die zusätzliche Festsetzung 4 (S. 197) $v_r(s_r) = 0$ ausserhalb des Kreises K (d.h. nichts anderes als dass es für die Längen bei gradlinigem Flug eine obere Schranke K gibt) gewährleistet ist. Ausserdem könnte man sich die Geraden G_r ausserhalb des Kreises K in geschlossene Kurven übergehend denken, ohne an dem Problem etwas zu ändern.

Ich möchte mich deshalb hier auf den prinzipiellen Gang der Herleitung beschränken. Einzelheiten können aus der zitierten Arbeit entnommen werden. Gefragt ist zunächst nach der Wahrscheinlichkeit, dass die Vektorsumme

$$y_1(s_1) + y_2(s_2) + \ldots + y_m(s_m)$$

zwischen y und $y + dy$ liegt, wenn der Endpunkt von $y_r(s_r)$ auf G_r gemäss der Wahrscheinlichkeitsdichte $v_r(s_r)$ variiert.

Wenn $$y = \sqrt{m}\,u + b_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

ist die Wahrscheinlichkeit dafür, dass gilt

$$\sqrt{m}\,u + b_m \leqslant \sum_{r=1}^{m} y_r(s_r) \leqslant \sqrt{m}\,(u + du) + b_m \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

gegeben durch

$$J_m(u) = \int_{(m)} \ldots \int v_1(s_1)\, v_2(s_2) \ldots v_m(s_m)\, ds_1 ds_2 \ldots ds_m \quad \ldots\ldots\ldots\ldots\ldots (3),$$

das m-fache Integral erstreckt über den Teil eines m-dimensionalen Würfels in dem (2) gilt.

Man setzt nun

$$y_1(s_1)+y_2(s_2)+\ldots+y_m(s_m)=\sqrt{m}\,u+b_m, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

wodurch die neue Veränderliche u auf b_m (Mittelwert) als Koordinatenursprung bezogen ist und der Massstab wie $1:\sqrt{m}$ verkleinert ist. Das besagt dasselbe, wie wenn ich im alten Massstab annehme, dass statt eines Fluges bestehend aus m Elementarflügen einer bestimmten Richtung G_r mit der resultierenden Verteilung $v_r(s_r)$, in diese Richtung nur einer dieser Elementarflüge ausgeführt wird, für den die Streuungskomponenten sind $s^{(r)}_{\lambda\mu}:m$ und dessen Mittelwert ist $a_r:\sqrt{m}$ mit den Komponenten $a_{r1}:\sqrt{m}$ und $a_{r2}:\sqrt{m}$ (s. (6) und (9)). Die durch (4) definierten zwei Gleichungen lassen sich nach s_{m-1} und s_m, wenn die entsprechende Funktionaldeterminante $D\neq 0$ ist, auflösen. Die Werte für s_{m-1} und s_m kann ich in (3) einsetzen und erhalte:

$$J_m(u)=\int\ldots\int v_1(s_1)\ldots v_{m-2}(s_{m-2})\,v_{m-1}(s_{m-1}[u])\,v_m(s_m[u]) \times\left|\frac{1}{D}\right| ds_1\ldots ds_{m-2}du_1du_2{}^{*} \quad \ldots\ldots(3a)$$

erstreckt über das m-dimensionale der Transformation entsprechende Integrationsgebiet. Dabei ist die Integration nach du_1 und du_2 gerade über das Intervall du_1du_2 erstreckt. Wenn

$$(u+du)\sqrt{m}+b_m \text{ in } u\sqrt{m}+b_m$$

übergeht, kommen wir von der Wahrscheinlichkeit $J_m(u)$ zur Wahrscheinlichkeitsdichte $w_m(u)$, indem wir bilden:

$$\lim_{\substack{du_1\to 0\\ du_2\to 0}}\frac{J_m}{du_1du_2}=w_m(u) =\int_{(m-2)}\ldots\int v_1(s_1)\ldots v_{m-2}(s_{m-2})\,v_{m-1}(s_{m-1}[u])\,v_m(s_m[u]) \times\left|\frac{1}{D}\right| ds_1\ldots ds_{m-2} \quad \ldots\ldots(5).$$

Dieses $(m-2)$-fache Integral multipliziert mit N geht nun für grosse m in unser gesuchtes Φ_1 über. Es ist also $\lim_{m\to\infty} w_m(u)$ zu untersuchen.

Es sei nun noch zu den $v_r(s_r)$ ein Mittelwert definiert als:

$$a_r=\int y_r v_r(s_r)\,ds_r{}^{\dagger} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

und es sei:

$$\sum_{r=1}^{m} a_r=b_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

und als "Streuung" die zweireihige symmetrische Matrix

$$(s^{(r)}_{\lambda\mu})=\begin{pmatrix} s^{(r)}_{11} & s^{(r)}_{12}\\ s^{(r)}_{21} & s^{(r)}_{22}\end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

* s_{m-1} und s_m hängen natürlich auch noch von s_1 bis s_{m-2} ab.

† Es soll im folgenden heissen:

$$\int=\int_{-\infty}^{+\infty},\qquad \int_a=\int_a^{+\infty},\qquad \int^b=\int_{-\infty}^{b}.$$

wobei $$s_{\lambda\mu}^{(r)} = \int v_r (y_{\lambda r} - a_{\lambda r})(y_{\mu r} - a_{\mu r})\, ds_r \quad \ldots\ldots(9),$$

und es sei: $$|a_r| < a, \quad |s_{\lambda\mu}^{(r)}| < s \quad \ldots\ldots(10),$$

$$\sum_{r=1}^{m} s_{\lambda\mu}^{(r)} = t_{\lambda\mu}^{(m)} = m h_{\lambda\mu}^{(m)} \quad \ldots\ldots(11),$$

also $$h_{\lambda\mu}^{(m)} = t_{\lambda\mu}^{(m)} : m \text{ und } \lim_{m \to \infty} h_{\lambda\mu}^{(m)} = h_{\lambda\mu} \quad \ldots\ldots(12).$$

Die $h_{\lambda\mu}$ seien endlich, sie bilden die positive definite Matrix der reduzierten Streuungen. Die Summe der Streuungskomponenten der $v_r(s_r)$ wachse also wie m ins Unendliche, d.h. die Streuungskomponenten der verschiedenen Richtungen seien fast alle von der gleichen Grössenordnung.

Zu v_r sei eine komplexe Adjunkte

$$f_r(u) = \int e^{i(u - a_r)(y_r(t) - a_r)}\, v_r(t)\, dt^* \quad \ldots\ldots(13)$$

definiert, sodass $$f_r(a_r) = 1 \quad \ldots\ldots(14),$$

$$\left(\frac{\partial f_r}{\partial s_\lambda}\right)_{a_r} = 0 \quad \ldots\ldots(15),$$

$$\left(\frac{\partial^2 f_r}{\partial s_\lambda \partial s_\mu}\right)_{a_r} = -s_{\lambda\mu}^{(r)}; \quad \lambda, \mu = 1, 2 \quad \ldots\ldots(16).$$

Analog sei die komplexe Adjunkte von w_m:

$$p_m(u) = \iint e^{izu} w_m(z)\, dz_1\, dz_2 \quad \ldots\ldots(17).$$

Führe ich nun statt z_1, z_2 die alten Variablen s_{m-1}, s_m ein, nämlich

$$z_\lambda = \frac{1}{\sqrt{m}} \sum_{r=1}^{m} (y_{\lambda r} - a_{\lambda r}); \quad \lambda = 1, 2 \quad \ldots\ldots(18),$$

so wird unter Berücksichtigung von (5),

$$p_m(u) = \int_{(m)} \ldots \int D\, \frac{1}{D}\, e^{\frac{i}{\sqrt{m}} \left\{ u_1 \sum_{r=1}^{m} (y_{1r} - a_{1r}) + u_2 \sum_{r=1}^{m} (y_{2r} - a_{2r}) \right\}} v_1(s_1) \ldots v_m(s_m)\, ds_1 \ldots ds_m$$

$$= \int e^{\frac{i}{\sqrt{m}} u (y_1 - a_1)} v_1(s_1)\, ds_1 \ldots \int e^{\frac{i}{\sqrt{m}} u (y_m - a_m)} v_m(s_m)\, ds_m$$

$$= f_1\left(a_1 + \frac{u}{\sqrt{m}}\right) \ldots f_m\left(a_m + \frac{u}{\sqrt{m}}\right) \quad \ldots\ldots(19),$$

d.h. die komplexe Adjunkte von $w_m(u)$ (ein Integralprodukt) ist auf ein gewöhn-

* ab bedeutet skalares Produkt der beiden Vektoren a und b. Also $ab = a_1 b_1 + a_2 b_2$, wobei a_1, a_2, b_1, b_2 die Komponenten von a bzw. b sind. In obigem Produkt entspricht: a dem Vektor $(u - a_r)$ und b dem Vektor $y_r(t) - a_r$).

liches Produkt der komplexen Adjunkten der einzelnen Verteilungen $v_r\left(a_r+\frac{u}{\sqrt{m}}\right)$ zurückgeführt. Nach einem Satz über gewöhnliche Produkte von v. Mises (s. Lit.-Verz. Nr. (8)) gilt nun unter gewissen Voraussetzungen, die hier zutreffen, wenn für die Funktionen f_r (14) bis (16) gilt:

$$\lim_{m\to\infty} f_1\left(a_1+\frac{u}{\sqrt{m}}\right) f_2\left(a_2+\frac{u}{\sqrt{m}}\right)\dots f_m\left(a_m+\frac{u}{\sqrt{m}}\right)$$
$$= \lim_{m\to\infty} p_m(u)$$
$$= e^{-\frac{1}{2}(h_{11}u_1^2-2h_{12}u_1u_2+h_{22}u_2^2)}, \quad \dots\dots\dots\dots(20),$$

$h_{\lambda\mu}$ berechnet sich aus den $s^{(r)}_{\lambda\mu}$ nach (11), (12); ausserdem gilt:

$$\lim_{m\to\infty}\iint p_m(u)\,\psi(u)\,du_1du_2=\iint e^{-\frac{1}{2}\sum_{\lambda\mu}h_{\lambda\mu}u_\lambda u_\mu}\psi(u)\,du_1du_2 \quad \dots\dots(21).$$

Mit Hilfe des Fourier'schen Integraltheorems lässt sich nun umgekehrt aus der Adjunkte einer Verteilung diese selbst wiedergewinnen und zwar ist:

$$w_m(u)=\left(\frac{1}{2\pi}\right)^2\iint e^{-izu}p_m(z)\,dz_1dz_2 \quad \dots\dots\dots\dots(22),$$

wobei p_m die Adjunkte von w_m ist.

Also bei Benutzung der vorigen Formel (21) mit $\psi(u)=e^{-izu}$ wird:

$$\lim_{m\to\infty} w_m(u)=\lim_{m\to\infty}\left(\frac{1}{2\pi}\right)^2\iint e^{-izu}p_m(z)\,dz_1dz_2$$
$$=\left(\frac{1}{2\pi}\right)^2\iint e^{-izu}e^{-\frac{1}{2}\sum_{\lambda\mu}h_{\lambda\mu}z_\lambda z_\mu}dz_1dz_2 \quad \dots\dots\dots\dots(23).$$

Das Doppelintegral lässt sich auswerten durch zweimalige Anwendung der Formel

$$\int e^{-pz^2-2qz}dz=\sqrt{\frac{\pi}{p}}\,e^{\frac{q^2}{p}} \quad \dots\dots\dots\dots(24),$$

wobei ich erhalte:

$$\lim_{m\to\infty} w_m(u)=w(u)=\frac{1}{2\pi\sqrt{h_{11}h_{22}-h_{12}^2}}\,e^{-\frac{h_{22}u_1^2-2h_{12}u_1u_2+h_{11}u_2^2}{2(h_{11}h_{22}-h_{12}^2)}} \quad \dots\dots(25).$$

Das wäre also die Wahrscheinlichkeitsdichte für den Punkt P, den Endpunkt des Vektors $u\sqrt{m}+b$, den man durch die angegebene Vektoraddition erhält. Oder bei der oben auf Seite 198 gemachten Annahme, die ja im wesentlichen nichts anderes besagt, als dass die Länge einer gradlinigen Flugstrecke umgekehrt proportional ist der Anzahl der überhaupt ausgeführten gradlinigen Flugstrecken, ist

$$Nw(u)=\Phi_1(u) \quad \dots\dots\dots\dots(26).$$

Das ist die Dichte der Mücken—also Anzahl pro Flächeneinheit—nach dem ersten Flug an einem Punkt P dem Endpunkt des von Q abgetragenen Vektors u. (Dabei

ist $\lim_{m\to\infty} \frac{b_m}{\sqrt{m}} = b = 0$ angenommen, andernfalls ist die gesammte Verteilung um den Vektor b zu verschieben.)

Als Lösung des Grundproblems erhalten wir also eine Gauss'sche Verteilung

$$\Phi_1(u) = \frac{N}{2\pi\sqrt{h_{11}h_{22} - h_{12}^2}} e^{-\frac{h_{22}u_1^2 - 2h_{12}u_1u_2 + h_{11}u_2^2}{2(h_{11}h_{22} - h_{12}^2)}} \quad \text{.....(27).}$$

Durch geeignete Wahl des Koordinatensystem, Drehung um den Winkel

$$\phi = \tfrac{1}{2} \operatorname{arc} tg \frac{2h_{12}}{h_{11} - h_{22}},$$

mittels orthogonaler Transformation der Koordinaten, indem man setzt:

$$u_1 = x\cos\phi - y\sin\phi, \quad u_2 = x\sin\phi + y\cos\phi,$$

lässt sich (27) noch vereinfachen, und man erhält:

$$\Phi_1(u) = \frac{N}{2\pi s_1 s_2} e^{-\frac{x^2}{2s_1^2} - \frac{y^2}{2s_2^2}} \quad \text{.....(28),}$$

dabei ist $$s_1^2 = \frac{2(h_{11}h_{22} - h_{12}^2)}{h_{11} + h_{22} + \sqrt{4h_{12}^2 + (h_{22} - h_{11})^2}},$$

und $$s_2^2 = \frac{2(h_{11}h_{22} - h_{12}^2)}{h_{11} + h_{22} - \sqrt{4h_{12}^2 + (h_{22} - h_{11})^2}}.$$

Sind nun keine Teile der Ebene, über die sich die Verteilung erstreckt, irgendwie ausgezeichnet, was ich hier der Einfachheit halber annehmen möchte, so muss aus Symmetriegründen $s_1^2 = s_2^2$ sein, und es wird:

$$\left.\begin{aligned} \Phi_1(r) &= \frac{N}{2\pi s^2} e^{-\frac{r^2}{2s^2}} \qquad (r^2 = x^2 + y^2) \\ &= \frac{N}{\pi\sigma^2} e^{-\frac{r^2}{\sigma^2}} \end{aligned}\right\} \quad \text{.....(29).}$$

$2s^2 = \sigma^2$ kann man als "Streuung" des Abstandes r vom Ursprung bezeichnen, wenn man als Definition der Streuung ansieht:

$$\sigma^2 = \frac{1}{N}\sum_\nu r_\nu^2 \Phi(r_\nu) f_\nu,$$

oder bei geometrischer Verteilung:

$$\begin{aligned} \sigma^2 &= \int_0^{2\pi}\int_0^\infty \frac{1}{2\pi s^2} e^{-\frac{r^2}{2s^2}} r^3\, dr\, d\phi = \frac{1}{2\pi s^2}\iint (x^2 + y^2)\, e^{-\frac{x^2+y^2}{2s^2}}\, dx\, dy \\ &= \frac{1}{\sqrt{2\pi}\, s}\int\left\{ e^{-\frac{y^2}{2s^2}} \frac{1}{\sqrt{2\pi}\, s}\int x^2 e^{-\frac{x^2}{2s^2}}\, dx\right\} dy \\ &\quad + \frac{1}{\sqrt{2\pi}\, s}\int\left\{ e^{-\frac{x^2}{2s^2}} \frac{1}{\sqrt{2\pi}\, s}\int y^2 e^{-\frac{y^2}{2s^2}}\, dy\right\} dx \\ &= s^2 + s^2 = 2s^2 = \sigma^2 \quad \text{.....(30),} \end{aligned}$$

da allgemein gilt (s. Formel (24)):

$$\frac{1}{\sqrt{2\pi}s}\int x^2 e^{-\frac{x^2}{2s^2}}dx = s^2, \quad \frac{1}{\sqrt{2\pi}s}\int e^{-\frac{x^2}{2s^2}}dx = 1.$$

Analog ist der mittlere oder durchschnittliche Abstand von Q:

$$l = \frac{1}{N}\int_0^{2\pi}\int_0 r\Phi_1(r)\,r\,dr\,d\phi = \frac{1}{2\pi s^2}\,2\pi\int_0 r^2 e^{-\frac{r^2}{2s^2}}dr = \sqrt{\frac{\pi}{2}}\,s = \frac{\sigma}{2}\sqrt{\pi} \ldots (31).$$

Das wäre wohl die Grösse, die in den früheren Arbeiten als durchschnittliche Fluglänge l bezeichnet wird, obgleich es genauer heissen müsste: durchschnittlicher Abstand nach einem Fluge von einem Punkt aus, denn die im Durchschnitt tatsächlich durchflogene Strecke ist zweifellos grösser.

3. *Die von einem Punkt ausgehende Verteilung nach dem n-ten Fluge.*

Nach dem ersten Flug befinden sich auf einem Flächenelement $dx'dy'$ mit dem inneren Punkt $\overline{Q}_{x',y'}$ $\frac{N}{\pi\sigma^2}e^{-\frac{r'^2}{\sigma^2}}dx'dy'$ Mücken. Beim nächsten Flug verteilen sich nun diese von $\overline{Q}$ aus wie vorher die Gesamtzahl N von Q aus. Der entsprechende Vorgang findet von sämtlichen Flächenelementen der Ebene statt. Alle diese Verteilungen überlagern sich und ergeben die Gesamtverteilung nach dem zweiten Flug. Entsprechend erhält man alle weiteren Verteilungen. Es sei nun

$$\Phi_{n-1}(r) = \frac{N}{\pi(n-1)\sigma^2}e^{-\frac{r^2}{(n-1)\sigma^2}} \ldots\ldots\ldots\ldots\ldots\ldots (32),$$

dann ergibt sich nach der Methode der vollständigen Induktion, da (32) für $n = 2$ richtig ist, nämlich

$$\Phi_1(r) = \frac{N}{\pi\sigma^2}e^{-\frac{r^2}{\sigma^2}} \ldots\ldots\ldots\ldots\ldots\ldots (29),$$

$$\Phi_n(r) = \iint \frac{N}{\pi(n-1)\sigma^2}e^{-\frac{r'^2}{(n-1)\sigma^2}}dx'dy'\,\frac{1}{\pi\sigma^2}e^{-\frac{(r-r')^2}{\sigma^2}}$$

$$= \frac{N}{\pi^2(n-1)\sigma^4}e^{-\frac{r^2}{\sigma^2}}\iint e^{-\frac{nr'^2-2(n-1)rr'}{(n-1)\sigma^2}}dx'dy'$$

$$= \frac{N}{\pi^2(n-1)\sigma^4}e^{-\frac{r^2}{\sigma^2}}\int\left\{e^{-\frac{2(n-1)xx'+nx'^2}{(n-1)\sigma^2}}\int e^{-\frac{2(n-1)yy'+ny'^2}{(n-1)\sigma^2}}dy'\right\}dx',$$

nach zweimaliger Anwendung der Formel (24) erhält man:

$$= \frac{N}{\pi^2(n-1)\sigma^4}e^{-\frac{r^2}{\sigma^2}}\sqrt{\frac{\pi(n-1)\sigma^2}{n}}\,e^{\frac{(n-1)\sigma^2y^2}{n}}\sqrt{\frac{\pi(n-1)\sigma^2}{n}}\,e^{\frac{(n-1)\sigma^2x^2}{n}}$$

$$= \frac{N}{n\pi\sigma^2}e^{-\frac{r^2}{n\sigma^2}} = \frac{N}{4nl^2}e^{-\frac{r^2\pi}{4nl^2}} = \Phi_n(r) \ldots\ldots\ldots\ldots\ldots\ldots (33).$$

Die Rayleigh-Pearson'sche Lösung war für $n \to \infty$

$$\Phi_n(r) = \frac{N}{n\pi l^2} e^{-\frac{r^2}{nl^2}} \quad \text{.....(V).}$$

Sie weicht also für grosse n nur in geringem Masse von der hier erhaltenen ab und würde mit dieser zusammenfallen, wenn mit l^2 die "Streuung" nach dem ersten Flug gemeint wäre.

In einem Kreis um Q mit dem Radius σ befinden sich nach n Flügen:

$$W_n(\sigma) = \int_0^{\sigma}\int_0^{2\pi} \frac{N}{n\pi\sigma^2} e^{-\frac{r^2}{n\sigma^2}} r\,dr\,d\phi = N\left[-e^{-\frac{r^2}{n\sigma^2}}\right]_0^{\sigma}$$

$$= N\left(1 - e^{-\frac{1}{n}}\right) = N\left(\frac{1}{n} - \frac{1}{2!\,n^2} + \frac{1}{3!\,n^3} - \ldots\right) \quad \text{.....(34).}$$

Das stimmt in erster Näherung mit dem Resultat von Kluyver (*l.c.* S. 344) bzw. Pearson überein, wenn wie oben mit der mittleren Fluglänge l das σ gemeint ist. Sie erhielten:

$$W_n(\sigma) = N:(n+1) = N\left(\frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^3} - \frac{1}{n^4} + \ldots\right) \quad \text{.....(34 a).}$$

Die Verteilung nach m Flugperioden, deren jede aus n Flügen besteht, ergibt sich ohne weiteres aus (29), indem man setzt:

$$n\sigma^2 \text{ statt } \sigma^2 \quad \text{und} \quad m \text{ statt } n.$$

Es wird dann nach (33)

$${}_m\Phi_n(r) = \Phi_{mn}(r) \quad \text{.....(35).}$$

Also resultiert die gleiche Verteilung, ob m-mal Flugperioden aus je n Flügen stattfinden oder ob eine Flugperiode aus mn Flügen stattfindet.

Ist nun von einer Mückenart M_1 das σ_1 bekannt, so lässt sich daraus für eine andere $M_2\sigma_2$ berechnen, wenn man auch für diese nach dem sogenannten Prinzip von "Trial and Error" in der hier ausgeführten Weise die Verteilung, durch "alltägliche Flüge" (s. Lit.-Verz. Nr. (17), S. 237) entstehend, annehmen kann, dass also nach der in der Biologie üblichen Ausdrucksweise "Random Migration" vorliegt. Zunächst sei die Dauer eines Fluges von M_2 t-mal so gross wie die von M_1. Dann kann man die Verteilung von M_2 nach einer Flugperiode auffassen als die Verteilung von M_1 nach t Flugperioden. Oder man nimmt an, dass bei t-facher Flugdauer statt eines Elementarfluges einer bestimmten Richtung G_r t Elementarflüge dieser Art ausgeführt werden, wodurch die Streuungskomponenten $s_{\lambda\mu}^{(r)}$ mit t zu multiplizieren sind und daher auch, wie man aus Formel (8) bis (12), (28) und (30) sieht:

$$\sigma_2^2 = t\sigma_1^2 \quad \text{.....(36).}$$

Also sowohl nach (29) in Verbindung mit (33) wie nach (35) erhält man:

$${}^{(2)}\Phi_n(r) = {}^{(1)}\Phi_{tn}(r) \quad \text{.....(37).}$$

Andererseits sei die lineare Fluggeschwindigkeit von M_2 k-mal so gross wie die von M_1. Dann wird man annehmen können, dass die Wahrscheinlichkeit für

die Länge eines Fluges längs G_r bei M_1 gleich ist der für die k-fache Länge bei M_2 längs derselben Geraden G_r. Man hat also in (6) bis (9) für y_r ky_r zu setzen. Daraus folgt, dass $s_{\lambda\mu}^{(r)}$ in $k^2 s_{\lambda\mu}^{(r)}$ übergeht, und nach Berücksichtigung der gleichen Formeln wie oben wird also

$$\sigma_2^2 = k^2 \sigma_1^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38).$$

Unterscheidet sich also M_2 von M_1 durch die t-fache Flugdauer und die k-fache Fluggeschwindigkeit, so erhält man die Verteilung für M_2 aus der für M_1 als:

$$^{(2)}\Phi_n(r) = {}^{(1)}\Phi_{tk^2 n}(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39).$$

Umgekehrt lässt sich von den Verteilungen zweier Tierarten auf das Verhältnis ihrer Fluggeschwindigkeiten schliessen, eine Grösse, die bei Untersuchungen von A. J. Lotka über das Fangen und Entkommen der Beute eines Raubtieres von grundlegender Bedeutung sein dürfte (s. Lit.-Verz. Nr. (21)).

4. *Verteilung von einem mit der Dichte N besetzten Gebiet in die unbesetzte Umgebung**.

Es gehe jetzt die Verteilung von einer irgendwie begrenzten und gleichmässig dicht besetzten Fläche Q aus. Die Dichte sei N. Es befinden sich also in einem Flächenelement $dx'dy'$ $N dx'dy'$ Mücken. Die Verteilung von jedem Flächenelement aus sei gleichmässig nach allen Seiten wie in den §§ 2 und 3. Dann wird nach n Flügen die Dichte an einem Punkt $P = (x, y)$ sein:

$$F_n(x, y) = \iint_{(Q)} N dx'dy' \frac{1}{N} \Phi_n\left(\sqrt{(x-x')^2 + (y-y')^2}\right)$$

$$= \iint_{(Q)} A\, df \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40).$$

Das Integral ist über die Fläche Q zu erstrecken.

Sei andererseits für das Äussere eines Gebietes Q die Dichte gleich N, so ergibt sich für die Verteilung in das ursprünglich freie Innere nach dem n-ten Flug:

$$\overline{F}_n(x, y) = \iint_{(Qa)} A df = \iint A df - \iint_{(Qi)} A df = N - F_n(x, y) \quad \ldots\ldots (41),$$

denn das Integral über das Äussere einer Fläche Q ist gleich dem Integral über die ganze Ebene vermindert um das Integral über das Innere von Q.

Beispiele für verschiedene Gebiete.

(a) Verteilung von einer durch eine gerade Linie L begrenzten Halbebene aus.

Die Dichte rechts von L sei N. Wie gross ist nach n Flügen die Dichte F im Punkt O (Koordinatenanfangspunkt) mit dem Abstand $-x$ von L?

* Teilweise sind die mathematischen Formeln dieses Paragraphen im Prinzip in der Pearson'schen Arbeit (s. Lit.-Verz. Nr. (3)) schon enthalten, da sie aus den dortigen komplizierteren Sätzen jedoch nicht so ohne weiteres ersichtlich sind und ich häufig eine etwas andere Ableitung gewählt habe, habe ich nähere Hinweise unterlassen.

Nach (40), wenn die y-Achse parallel L, ist

$$F_n(x) = F_n(0,0) = \int_x \int \Phi_n(\sqrt{x'^2+y'^2})\,dx'dy'$$

$$= \frac{N}{n\pi\sigma^2}\int_x \int e^{-\frac{x'^2+y'^2}{n\sigma^2}}\,dx'dy'$$

$$= \frac{N}{n\pi\sigma^2}\int_x \left\{e^{-\frac{x'^2}{n\sigma^2}}\int e^{-\frac{y'^2}{n\sigma^2}}dy'\right\}dx'$$

$$= \frac{N}{\sqrt{n\pi}\,\sigma}\int_x e^{-\frac{x'^2}{n\sigma^2}}\,dx'$$

$$= \frac{N}{\sqrt{\pi n}\,\sigma}\left\{\int e^{-\frac{x'^2}{n\sigma^2}}dx' - \int^x e^{-\frac{x'^2}{n\sigma^2}}dx'\right\}$$

$$= N\left\{1 - \frac{1}{\sqrt{\pi n}\,\sigma}\int^x e^{-\frac{x^2}{n\sigma^2}}dx\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(42),$$

d.i. also die Dichte an einem Punkt mit dem Abstand $-x$ von L. Die Werte des Integrals sind aus Tabellen, wie sie fast in jedem Werk über Wahrscheinlichkeitsrechnung oder Statistik (z.B. Lit.-Verz. Nr. (9) bis (11)) enthalten sind, zu entnehmen. Für $x=0$, d.h. auf der Grenzlinie L ist die Dichte wie zu erwarten gleich $\frac{1}{2}N$.

(b) Verteilung von einem Rechteck mit den Seiten $2a$ und $2b$.

Die Dichte sei wieder N. Die Koordinatenachsen sein parallel den Rechteckseiten und ihr Anfangspunkt sei der Schnittpunkt der Diagonalen. Dann ist die Dichte nach dem n-ten Flug im Punkt (x, y):

$$F_n(x,y) = \int_{-a}^{+a}\int_{-b}^{+b} \Phi_n(\sqrt{(x'-x)^2+(y'-y)^2})\,dx'dy'$$

$$= \frac{N}{\pi n\sigma^2}\int_{-a}^{a}\int_{-b}^{b} e^{-\frac{(x'-x)^2+(y'-y)^2}{n\sigma^2}}\,dx'\,dy'$$

$$= \frac{N}{\sqrt{\pi n}\,\sigma}\int_{-a}^{+a} e^{-\frac{(x'-x)^2}{n\sigma^2}}\,dx' \frac{1}{\sqrt{\pi n}\,\sigma}\int_{-b}^{+b} e^{-\frac{(y'-y)^2}{n\sigma^2}}\,dy'$$

$$= N\left\{1 - \int^{-a-x}\frac{1}{\sqrt{n\pi}\,\sigma}\,e^{-\frac{x'^2}{n\sigma^2}}dx' - \int^{-a+x}\frac{1}{\sqrt{n\pi}\,\sigma}\,e^{-\frac{x'^2}{n\sigma^2}}dx'\right\}$$

$$\times\left\{1 - \int^{-b-y}(\ldots y'\ldots) - \int^{-b+y}(\ldots y'\ldots)\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(43).$$

Bei der Verteilung von einem schmalen Streifen, d.h. $b=\infty$, wird der zweite Faktor $\{1\}$.

(c) Verteilung von einem Kreis mit dem Radius a um O als Mittelpunkt.

Die Dichte sei wieder N. Dann ist nach n Flügen die Dichte an einem Punkt im Abstand c von O:

$$F_n(c) = \int_0^a\int_0^{2\pi} \Phi_n(\sqrt{c^2+r^2-2rc\cos\theta})\,r\,dr\,d\theta$$

$$= \frac{N}{\pi n\sigma^2}\int_0^a\int_0^{2\pi} e^{-\frac{c^2+r^2-2rc\cos\theta}{n\sigma^2}}\,r\,dr\,d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(44),$$

nach Entwicklung von $e^{\frac{2rc\cos\theta}{n\sigma^2}}$ in die bekannte e-Reihe:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \dots\dots\dots\dots(45),$$

wird also $F_n(c) = \frac{N}{\pi n\sigma^2} e^{-\frac{c^2}{n\sigma^2}} \int_0^a \int_0^{2\pi} \left\{ e^{-\frac{r^2}{n\sigma^2}} r \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{2rc}{n\sigma^2}\right)^k \cos^k\theta \right\} d\theta\, dr.$

Da nun $\int_0^{2\pi} \cos^{2k+1}\theta\, d\theta = 0, \quad \int_0^{2\pi} \cos^{2k}\theta\, d\theta = \frac{2\pi}{2^{2k}} \binom{2k}{k} \dots\dots\dots\dots(46),$

wird weiter

$$F_n(c) = \frac{N}{\pi n\sigma^2} e^{-\frac{c^2}{n\sigma^2}} \int_0^a \left\{ e^{-\frac{r^2}{n\sigma^2}} r \sum_{k=0}^{\infty} \left(\frac{2rc}{n\sigma^2}\right)^{2k} \frac{2\pi}{(2^k k!)^2} \right\} dr$$

$$= \frac{2N}{n\sigma^2} e^{-\frac{c^2}{n\sigma^2}} \sum_{k=0}^{\infty} \left\{ \left(\frac{c}{n\sigma^2}\right)^{2k} \left(\frac{1}{k!}\right)^2 \int_0^a e^{-\frac{r^2}{n\sigma^2}} r^{2k+1} dr \right\}.$$

Die Integrale lassen sich sukzessiv durch Produktintegration auswerten, sodass also:

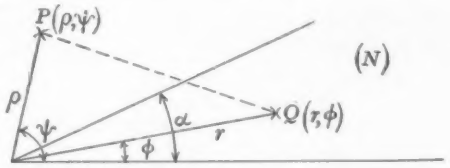
$$F_n(c) = N \left\{ 1 - e^{-\frac{c^2+a^2}{n\sigma^2}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{c^2}{n\sigma^2}\right)^k \left[1 + \frac{a^2}{n\sigma^2} + \frac{1}{2!}\left(\frac{a^2}{n\sigma^2}\right)^2 + \dots + \frac{1}{k!}\left(\frac{a^2}{n\sigma^2}\right)^k \right] \right\}$$

$\dots\dots\dots(47).$

Die Konvergenz dieser Summe ist bei $c^2 \ll n\sigma^2$ verhältnismässig gut. Man kann aber auch zur Auswertung bei Benutzung der vorhergehenden Form Tabellen über die höheren Momente der Gauss'schen Summenfunktion benutzen (s. Lit.-Verz. Nr. (9), S. 22, 23).

Ist bei den behandelten Verteilungen das Innere zunächst frei von Mücken, während ausserhalb der betrachteten Gebiete die Dichte N ist, so ergibt sich die Verteilung in das Gebiet hinein gemäss (41).

(*d*) Verteilung von einem Winkelraum aus.



Innerhalb des von den Schenkeln des Winkels α abgegrenzten Gebietes herrsche wieder eine Dichte N. Dann wird nach n Flügen an einem Punkt P mit den Polarkoordinaten ρ, ψ die Dichte sein:

$${}_nF_n(\rho,\psi) = \int_0^{\alpha}\int_0^{a} \Phi_n\left(\sqrt{r^2+\rho^2-2r\rho\cos(\psi-\phi)}\right) r\, dr\, d\phi \quad \dots\dots\dots(48)$$

— unter Berücksichtigung von (45) und $\theta = \psi - \phi$

$$= \frac{N}{\pi n\sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \int_0^{a} \left\{ e^{-\frac{r^2}{n\sigma^2}} r \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{2r\rho}{n\sigma^2}\right)^k \int_{\psi-\alpha}^{\psi} \cos^k\theta\, d\theta \right\} dr$$

$$= \frac{N}{\pi n\sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{2\rho}{n\sigma^2}\right)^k \int_0^{a} e^{-\frac{r^2}{n\sigma^2}} r^{k+1} dr \int_{\psi-\alpha}^{\psi} \cos^k\theta\, d\theta.$$

Wegen der unbedingten Konvergenz dieser Reihe kann man sie aufspalten und als Summe der Reihen mit geradem und ungeradem k schreiben. Berücksichtigt man dann noch, dass:

$$\left.\begin{aligned}\int_0 e^{-\frac{r^2}{n\sigma^2}} r^{2k+1}\,dr &= \frac{k!}{2}(n\sigma^2)^{k+1}\\ \int_0 e^{-\frac{r^2}{n\sigma^2}} r^{2(k+1)}\,dr &= \frac{(2k+1)!}{k!\,2^{2k+1}}\frac{\sqrt{\pi n}\,\sigma}{2}(n\sigma^2)^{k+1}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(49),$$

so wird:

$${}_aF_n(\rho,\psi) = Ne^{-\frac{\rho^2}{n\sigma^2}}\left\{\frac{1}{2\pi}\sum_{k=0}^{\infty}\left[\frac{k!}{(2k)!}\left(\frac{2\rho}{\sqrt{n}\sigma}\right)^{2k}\int_{\psi-\alpha}^{\psi}\cos^{2k}\theta\,d\theta\right]\right.$$
$$\left.+\frac{1}{2\sqrt{\pi}}\sum_{k=0}^{\infty}\left[\frac{1}{k!}\left(\frac{\rho}{\sqrt{n}\sigma}\right)^{2k+1}\int_{\psi-\alpha}^{\psi}\cos^{2k+1}\theta\,d\theta\right]\right\}.$$

Die noch verbleibenden Integrale lassen sich auch auswerten (s. Anmerkung*), und es wird nach Einsetzung und einfacher Umordnung:

$${}_aF_n(\rho,\psi) = N\left\{\frac{\alpha}{2\pi}+\frac{1}{\pi}e^{-\frac{\rho^2}{n\sigma^2}}\sum_{k=1}^{\infty}\left(\frac{\rho^2}{n\sigma^2}\right)^k\frac{k!}{(2k)!}\right.$$
$$\times\sum_{r=0}^{k-1}\binom{2k}{r}\frac{\cos([k-r][2\psi-\alpha])\sin[k-r]\alpha}{k-r}+\frac{\rho}{\sqrt{\pi n}\,\sigma}e^{-\frac{\rho^2}{n\sigma^2}}\sum_{k=0}^{\infty}\left(\frac{\rho^2}{n\sigma^2}\right)^k\frac{1}{2^{2k}k!}$$
$$\left.\times\sum_{r=0}^{k}\binom{2k+1}{r}\frac{\cos(2k+1-2r)\left(\psi-\frac{\alpha}{2}\right)\sin(2k+1-2r)\frac{\alpha}{2}}{2k+1-2r}\right\}\quad\ldots\ldots\ldots\ldots\ldots(50).$$

Die Güte der Konvergenz hängt wieder vor allem von $\frac{\rho^2}{n\sigma^2}$ ab.

Für $\alpha=\pi$, $\psi=\frac{1}{2}\pi$ erhält man, wie man sich durch Einsetzen und Reihenvergleichung überzeugen kann, den nach (42) zu erwartenden Wert*:

$$F_n(-\rho) = {}_aF_n\left(\rho,\frac{\pi}{2}\right).$$

* Es ist:
$$\cos^n x = (\tfrac{1}{2})^n(e^{ix}-e^{-ix})^n = (\tfrac{1}{2})^n\sum_{r=0}^{n}\binom{n}{r}e^{ix(n-2r)}$$
$$= (\tfrac{1}{2})^n\sum_{r=0}^{n}\binom{n}{r}\{\cos x(n-2r) - i\sin x(n-2r)\}.$$

Da $\cos^n x$ für reelle x reell ist, müssen die imaginären Glieder der Summe 0 ergeben, also:

$$\cos^n x = (\tfrac{1}{2})^n\sum_{r=0}^{n}\binom{n}{r}\cos x(n-2r),$$
$$\int_{\psi-\alpha}^{\psi}\cos^n x\,dx = (\tfrac{1}{2})^n\sum_{r=0}^{n}\binom{n}{r}\frac{\sin x(n-2r)}{n-2r}\Big|_{\psi-\alpha}^{\psi}$$
$$= (\tfrac{1}{2})^n\sum_{r=0}^{n}\binom{n}{r}\frac{2\cos(n-2r)\left|\left(\psi-\frac{\alpha}{2}\right)\right.\sin(n-2r)\frac{\alpha}{2}}{n-2r},$$

5. Ausbreitung in Gebiete, die durch Hindernisse (Gebirge, Wasser od. ähnl.) begrenzt sind.

Ich beschränke mich auf den Fall, dass die mögliche Ausbreitungsfläche durch eine gradlinige Grenze abgeschlossen ist und mache die Hypothese, dass ein von einem Punkt ausgehender Flug an dieser Geraden gespiegelt wird. Es wären zwar noch andere Möglichkeiten für das Verhalten der Mücken an dem Hindernis denkbar, mir erscheint jedoch bei der Annahme rein zufälliger Flüge die oben gemachte Voraussetzung am glaubwürdigsten.

(*a*) Verteilung der N Mücken von einem Zentrum aus.

Die Grenzgerade H verlaufe parallel der y-Achse im Abstand a von dieser. Von $Q=(0,0)$ aus mögen wieder N Mücken abfliegen. Dann ist nach dem ersten Flug die Dichte an einem Punkt (x, y) für $x \leqslant a$:

$$\frac{N}{\pi\sigma^2} e^{-\frac{x^2+y^2}{\sigma^2}},$$

dazu kommt die Dichte des Spiegelbildes von (x, y) bezüglich der Geraden H, nämlich:

$$\frac{N}{\pi\sigma^2} e^{-\frac{(x+2[a-x])^2+y^2}{\sigma^2}} = \frac{N}{\pi\sigma^2} e^{-\frac{(2a-x)^2+y^2}{\sigma^2}}.$$

Also wird die gesamte Dichte:

$$ {}_a\bar{\Phi}_1(x,y) = \frac{N}{\pi\sigma^2}\left(e^{-\frac{x^2+y^2}{\sigma^2}} + e^{-\frac{(2a-x)^2+y^2}{\sigma^2}}\right) \quad \text{.....(51).}$$

Die Gesamtzahl der Mücken ist natürlich wieder:

$$\int^{a}\int {}_a\bar{\Phi}_1(x,y)\,dx\,dy = N \quad \text{.....(52),}$$

wie bei der Verteilung (33), was man leicht nachrechnen kann.

Sei beim $(n-1)$-ten Flug jetzt die Verteilung gegeben durch ${}_a\bar{\Phi}_{n-1}(x,y)$, dann erhält man ${}_a\bar{\Phi}_n(x,y)$, indem von allen Flächenelementen $dx'\,dy'$ sich

$${}_a\bar{\Phi}_{n-1}(x',y')\,dx'\,dy' \text{ Mücken}$$

gemäss $\frac{1}{N}\,{}_{a-x'}\bar{\Phi}_1(x,y)$ verteilen, sich überlagern und ${}_a\bar{\Phi}_n(x,y)$ ergeben.

also
$$\int_{\psi-\alpha}^{\psi} \cos^{2k} x\,dx = (\tfrac{1}{2})^{2k}\binom{2k}{k}\alpha + \frac{2}{2^{2k}}\sum_{r=0}^{k-1}\binom{2k}{r}\frac{\cos(k-r)(2\psi-\alpha)\sin(k-r)\alpha}{k-r},$$

$$\int_{\psi-\alpha}^{\psi} \cos^{2k+1} x\,dx = \frac{2}{2^{2k}}\sum_{r=0}^{k}\binom{2k+1}{r}\frac{\cos(2k+1-2r)\left(\psi-\frac{\alpha}{2}\right)\sin(2k+1-2r)\frac{\alpha}{2}}{2k+1-2r}.$$

Für $\alpha=\pi$, $\psi=\frac{\pi}{2}$ erhält man die Formel

$$\frac{2^{2k}(k!)^2}{(2k+1)!} = \frac{1}{2^{2k}}\sum_{r=0}^{k}\binom{2k+1}{r}\frac{(-1)^{k-r}}{2k+1-2r},$$

da bekanntlich durch partielle Integration wird:

$$\int_{-\pi/2}^{+\pi/2}\cos^{2k+1}x\,dx = 2\int_{0}^{\pi/2}\cos^{2k+1}x\,dx = 2\,\frac{2^{2k}(k!)^2}{(2k+1)!}.$$

Die obige Formel ist beim Vergleich von (50) mit (42) anzuwenden.

Bei der Annahme, dass gilt:

für $x \leqslant a$: $\quad {}_a\bar{\Phi}_{n-1}(x,y) = \Phi_{n-1}(x,y) + \Phi_{n-1}(2a-x,y)$,

für $x > a$: $\quad {}_a\bar{\Phi}_{n-1}(x,y) = 0$,

folgt durch vollständige Induktion, da nach (51)

$$ {}_a\bar{\Phi}_1(x,y) = \Phi_1(x,y) + \Phi_1(2a-x,y) \quad \ldots\ldots\ldots\ldots\ldots\ldots (51\,a)$$

ist, dass gilt:

$$ {}_a\bar{\Phi}_n(x,y) = \int^a\!\!\int \left\{\Phi_{n-1}(x',y') + \Phi_{n-1}(2a-x',y')\right\} \frac{1}{N}$$

$$ \times \{\Phi_1(x-x',\, y-y') + \Phi_1(2a-x'-x,\, y-y')\}\, dx'\, dy' \quad \ldots\ldots (53).$$

Dies Integral lässt sich in die Summe aus vier Integralen aufspalten und zwar:

1. $$\int^a\!\!\int \Phi_{n-1}(x',y')\, \Phi_1(x-x',y-y')\, dx'\, dy'.$$

2. $$\int^a\!\!\int \Phi_{n-1}(2a-x',y')\, \Phi_1(2a-x'-x,y-y')\, dx'\, dy'$$

$$ = \int_a\!\!\int \Phi_{n-1}(\bar{x}',y')\, \Phi_1(\bar{x}'-x,y-y')\, d\bar{x}'\, dy',$$

wobei $$2a - x' = \bar{x}'.$$

Da nach (29) $\Phi_1(-x,y) = \Phi_1(+x,y)$, ergibt

1. + 2. $$= \iint \Phi_{n-1}(x',y')\, \Phi_1(x-x',y-y')\, dx'\, dy',$$

das ist nach (33) $= \Phi_n(x,y)$.

3. $$\int^a\!\!\int \Phi_{n-1}(x',y')\, \Phi_1(2a-x'-x,y-y')\, dx'\, dy',$$

4. $$\int^a\!\!\int \Phi_{n-1}(2a-x',y')\, \Phi_1(x-x',y-y')\, dx'\, dy'$$

$$ = \int_a\!\!\int \Phi_{n-1}(\bar{x}',y')\, \Phi_1(x-2a+\bar{x}',y-y')\, d\bar{x}'\, dy',$$

wobei $$2a - x' = \bar{x}',$$

also:

3. + 4. $$= \iint \Phi_{n-1}(x',y')\, \Phi_1(2a-x'-x,y-y')\, dx'\, dy',$$

das ist nach (33) $= \Phi_n(2a-x,y)$.

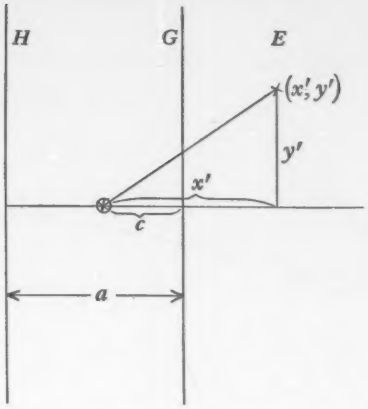
Also ist für $x \leqslant a$:

1. + 2. + 3. + 4. $\quad = {}_a\Phi_n(x,y) = \Phi_n(x,y) + \Phi_n(2a-x,y) \quad \ldots\ldots\ldots\ldots\ldots (53a)$,

w.z.b.w.

(*b*) Verteilung von einer mit der Dichte N besetzten Halbebene aus.

Die Verteilung gehe also von einer durch eine Gerade G begrenzten Halbebene E mit der Dichte N aus in ein Gebiet, das durch eine Gerade H, die parallel G sei, abgeschlossen ist.



Dann ist nach n Flügen an einem Punkt (dem Nullpunkt) im Abstand c von G die Dichte:

$$ {}_aF_n(c) = \int_c \int_{a+x'-c} \bar{\Phi}_n(x', y')\, dx'\, dy' $$

$$ = \int_c \int \Phi_n(x', y')\, dx'\, dy' + \int_c \int \Phi_n(2a - 2c + x', y')\, dx'\, dy' \quad \ldots\ldots(54), $$

das ergibt nach (42)

$$ = N\left\{1 - \frac{1}{\sqrt{\pi n}\,\sigma}\int^c e^{-\frac{x'^2}{n\sigma^2}} dx'\right\} $$

$$ + N\left\{1 - \frac{1}{\sqrt{\pi n}\,\sigma}\int^c e^{-\frac{(2a-2c+x')^2}{n\sigma^2}} dx'\right\} $$

$$ = N\left\{2 - \frac{1}{\sqrt{\pi n}\,\sigma}\left[\int^c e^{-\frac{x'^2}{n\sigma^2}} dx' + \int^{2a-c} e^{-\frac{x'^2}{n\sigma^2}} dx'\right]\right\} $$

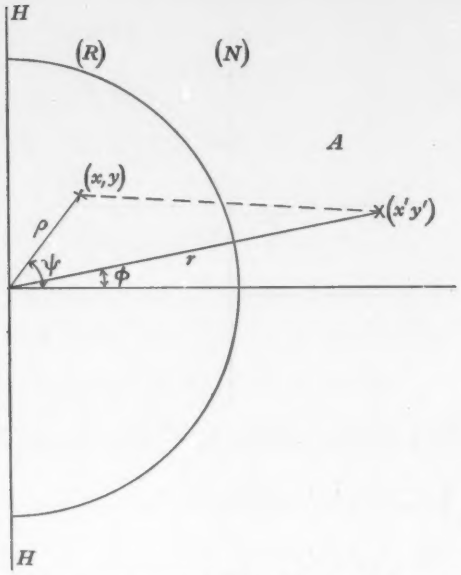
$$ = F_n(c) + F_n(2a - c). $$

Die Werte der Integrale lassen sich wieder aus Tabellen entnehmen (s. Lit.-Verz. Nr. (9) bis (11)).

Man kann also (54) erhalten, indem man zu der Halbebene E eine bezüglich H spiegelbildliche fiktive Halbebene E' mit der gleichen Dichte N annimmt.

Analog erhält man die Verteilung der anderen in § 4 besprochenen Gebiete, indem man immer das zu der jeweiligen Grenze H spiegelbildlich gelegene Gebiet mit der gleichen Dichte N hinzunimmt und die aus dem wirklichen und dem fiktiven Gebiet nach § 4 zu erwartenden Verteilungen addiert. Als Illustration dazu möchte ich noch die Verteilung in einen Halbkreis berechnen.

(c) Eine mit der Dichte N besetzte Halbebene sei wieder durch ein Hindernis H abgeschlossen. Gesucht ist die Verteilung in einen zunächst freien Halbkreis mit dem Radius R über H.



Nach dem n-ten Flug ist also an einem Punkt (x, y) (bzw. in Polarkoordinaten: ρ, ψ) die Dichte:

$$
\begin{aligned}
{}_R F_n(x,y) &= \int_{(A)} \int_{x'} \bar{\Phi}_n(x'-x, y'-y)\, dx'\, dy' \\
&= \int_{(A)} \int \Big\{ \Phi_n(x'-x, y'-y) + \Phi_n(x-x', y-y') \Big\}\, dx'\, dy' \quad \ldots\ldots(55).
\end{aligned}
$$

Das Integral ist über das ausserhalb des Halbkreises und rechts von H liegende Gebiet zu erstrecken. Durch Einführung von Polarkoordinaten:

$$x = \rho \cos\psi,\ y = \rho \sin\psi;\ \ x' = r\cos\phi,\ y' = r\sin\phi;$$

wird also:

$$
\begin{aligned}
{}_R F_n(\rho,\psi) = \int_R \int_{-\pi/2}^{+\pi/2} \frac{N}{\pi n\sigma^2} \Big\{ & e^{-\frac{r^2+\rho^2-2r\rho\cos(\phi-\psi)}{n\sigma^2}} \\
& + e^{-\frac{r^2+\rho^2+2r\rho\cos(\phi+\psi)}{n\sigma^2}} \Big\}\, r\, dr\, d\phi \ldots(56),
\end{aligned}
$$

nach (45)

$$
\begin{aligned}
= \frac{N}{\pi n\sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \int_R e^{-\frac{r^2}{n\sigma^2}} r \Big\{ & \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{2r\rho}{n\sigma^2}\right)^k \int_{-\pi/2}^{+\pi/2} \cos^k(\phi-\psi)\, d\phi \\
& + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{2r\rho}{n\sigma^2}\right)^k \int_{-\pi/2}^{+\pi/2} \cos^k(\phi+\psi)\, d\phi \Big\}\, dr.
\end{aligned}
$$

Nun ist aber:

$$
\begin{aligned}
\int_{-\pi/2}^{+\pi/2} \cos^k(\phi-\psi)\, d\phi = \int_{-\pi/2-\psi}^{+\pi/2-\psi} \cos^k\phi\, d\phi &= \int_{-\pi/2+\psi}^{+\pi/2+\psi} \cos^k\phi\, d\phi \\
&= \int_{-\pi/2}^{+\pi/2} \cos^k(\phi+\psi)\, d\phi \quad \ldots\ldots\ldots\ldots(57),
\end{aligned}
$$

folglich heben sich die in der Klammer $\{\ldots\ldots\}$ stehenden Glieder mit ungeradem k heraus, und es bleibt, da ausserdem:

$$\int_{-\pi/2}^{+\pi/2} \cos^{2k}(\phi \pm \psi)\, d\phi = \frac{\pi}{2^{2k}}\binom{2k}{k},$$

$$
\begin{aligned}
{}_RF_n(\rho,\psi) &= \frac{N}{\pi n\sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \int_R r e^{-\frac{r^2}{n\sigma^2}} 2 \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{2r\rho}{n\sigma^2}\right)^{2k} \frac{\pi}{2^{2k}}\binom{2k}{k} dr \\
&= \frac{2N}{n\sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{\rho}{n\sigma^2}\right)^{2k} \int_R e^{-\frac{r^2}{n\sigma^2}} r^{2k+1}\, dr \\
&= N\left\{1 - \frac{2}{n\sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{\rho}{n\sigma^2}\right)^{2k} \int_0^R e^{-\frac{r^2}{n\sigma^2}} r^{2k+1} dr\right\} \\
&= N - F_n(\rho) \quad \ldots\ldots\ldots\

also muss gelten:

$$\frac{1}{\sqrt{\pi}\,\sigma}\int\left\{e^{-\frac{(x-x')^2}{\sigma^2}}\frac{1}{\sqrt{\pi\,(n-1)}\,\sigma}\int^{-x'}e^{-\frac{x''^2}{(n-1)\,\sigma^2}}dx''\right\}dx'=\frac{1}{\sqrt{\pi n}\,\sigma}\int^{-x}e^{-\frac{x'^2}{n\sigma^2}}dx' \quad \ldots\ldots\ldots\ldots\ldots(60),$$

oder wenn ich setze $\frac{x''^2}{(n-1)\,\sigma^2}=\frac{\bar{x}''^2}{2}$ und $\left(\frac{x-x'}{\sigma}\right)^2=\frac{\bar{x}'^2}{2}$ und dann wieder $\bar{x}''$ mit x'' und $\bar{x}'$ mit x' bezeichne, wird:

$$\frac{1}{\sqrt{2\pi}}\int\left\{e^{-\frac{x'^2}{2}}\int^{\frac{x'}{\sqrt{n-1}}-\frac{x}{\sigma}\sqrt{\frac{2}{n-1}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx''\right\}dx'$$

$$=\int^{-\frac{x}{\sigma}\sqrt{\frac{2}{n}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx' \quad \ldots\ldots(61),$$

oder, wenn man setzt $\frac{1}{\sqrt{n+1}}=a$ und $-\frac{x}{\sigma}\sqrt{\frac{2}{n-1}}=b$,

$$\int\left\{\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}\int^{ax'+b}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx''\right\}dx'=\int^{b/\sqrt{1+a^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx' \ldots(62).$$

Weiter folgt daraus, dass gelten muss:

$$\int\left\{\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{x'^2}{2\sigma^2}}\int^{ax'+b}\frac{1}{\sqrt{2\pi}\,\rho}e^{-\frac{x''^2}{2\rho^2}}dx''\right\}dx'=\int^{b/\sqrt{\rho^2+\sigma^2a^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx'(63).$$

(*a*) Es soll jetzt die Verteilung von N Mücken ausgehend von einem Punkt (a, b) nach dem n-ten Flug bestimmt werden, wenn links der y-Achse der Vernichtungsfaktor $\Delta=\delta=$ konst. und rechts $\Delta=0$ ist, also für

$$x<0,\ \Delta=\delta;\quad x>0,\ \Delta=0.$$

Δ wird im allgemeinen natürlich durch Vermehrung und Absterben bestimmt und ist kleiner beziehungsweise grösser Null, je nachdem ob das erste oder zweite überwiegt.

Nach dem ersten Flug erhalten wir dann folgende Verteilung (nach (33))*:

$$\left.\begin{aligned}\bar{\Phi}_1(x-a,y-b)&=\Phi_1(x-a,y-b) \quad \text{für } x>0\\ &=(1-\delta)\,\Phi_1(x-a,y-b) \quad \text{für } x<0\end{aligned}\right\} \ldots\ldots\ldots\ldots(64).$$

Diese unstetige Funktion $\bar{\Phi}_1(x-a, y-b)$ kann man nun beliebig gut approximieren durch

$$\Phi_1(x-a,y-b)\left\{1-\delta\int^{-k_1x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx'\right\} \quad \ldots\ldots\ldots\ldots(65),$$

wenn nur k_1 genügend gross gewählt wird.

Ich setze also:

$$\bar{\Phi}_1(x-a,y-b)=\Phi_1(x-a,y-b)\left\{1-\delta\int^{-k_1x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx'\right\} \ldots(65^{\text{bis}}).$$

* Dabei ist allerdings angenommen, Vermehrung und Absterben finden immer am Ende eines Fluges statt.

Die Verteilung nach dem zweiten Flug wird dann bei Verwendung der noch abzuleitenden Formel (67):

$$\bar{\Phi}_2(x-a,y-b)$$

$$= \frac{1}{N}\iint \bar{\Phi}_1(x'-a,y'-b)\,\Phi_1(x-x',y-y')\,dx'\,dy'\left\{1-\delta\int^{-k_2 x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx'\right\}$$

$$= \Phi_2(x-a,y-b)\left\{1-\delta\int^{-(a+x)\sqrt{k_1^2/(4+k_1^2\sigma^2)}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx'\right\}$$

$$\times\left\{1-\delta\int^{-k_2 x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx'\right\} \quad \ldots\ldots(66),$$

oder bei $k_1 \to \infty$ wird:

$$\bar{\Phi}_2(x-a,y-b) = \Phi_2(x-a,y-b)\left\{1-\delta\left[\int^{-(a+x)/\sigma}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx' + \int^{-k_2 x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx'\right] + \delta^2\int^{-(a+x)/\sigma}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx'\int^{-k_2 x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx'\right\}$$

$\ldots\ldots(66\,a),$

dabei ist wieder $k_2 \to \infty$ zu nehmen.

Es gilt nämlich allgemein:

$$\frac{1}{N}\iint\left\{\Phi_n(x'-a,y'-b)\,\Phi_1(x-x',y-y')\int^{-(\alpha x'+\beta)}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx''\right\}dx'\,dy'$$

$$= N\iint\left\{\frac{1}{\pi^2 n^2\sigma^4}e^{-\frac{(x'-a)^2+(y'-b)^2}{n\sigma^2}-\frac{(x-x')^2+(y-y')^2}{\sigma^2}}\int^{-(\alpha x'+\beta)}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx''\right\}dx'\,dy'$$

$$= \frac{N}{\pi^2 n^2\sigma^4}\int e^{-\frac{(y'-b)^2+n(y-y')^2}{n\sigma^2}}dy'\int\left\{e^{-\frac{(x'-a)^2+n(x-x')^2}{n\sigma^2}}\int^{-(\alpha x'+\beta)}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx''\right\}dx'$$

$$= \frac{N}{\pi^2 n^2\sigma^4}\sqrt{\frac{\pi n}{n+1}}\,e^{-\frac{(y-b)^2+(x-a)^2}{(n+1)\sigma^2}}\int e^{-\frac{\left(x'-\frac{a+nx}{n+1}\right)^2}{\frac{n}{n+1}\sigma^2}}\int^{-(\alpha x'+\beta)}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx''\,dx'$$

$\ldots\ldots(67),$

wenn $x'-\dfrac{a+nx}{n+1}=\zeta,$

$$= \Phi_{n+1}(x-a,y-b)\,\frac{1}{\pi\sigma}\sqrt{\frac{\pi(n+1)}{n}}\int\left\{e^{-\frac{\zeta^2}{2\frac{n}{2(n+1)}\sigma^2}}\right.$$

$$\left.\times\int^{-\left(\zeta-\frac{a+nx}{n+1}\right)\alpha-\beta}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx''\right\}d\zeta,$$

daraus folgt unter Anwendung von (63):

$$= \Phi_{n+1}(x-a,y-b)\int^{-K}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx,$$

wobei

$$K = \frac{x n\alpha + a\alpha + (n+1)\beta}{\sqrt{\tfrac{1}{2}(n+1)\{2(n+1)+n\alpha^2\sigma^2\}}} \quad \ldots\ldots(68).$$

Bei Benutzung dieser Formeln folgt aus (66), (66 *a*), indem man setzt $n=1, \beta=0$, $\alpha=k_1\to\infty$.

Da im allgemeinen $\delta \ll 1$ sein wird, kann man, ohne einen grossen Fehler zu begehen, Glieder mit δ^2 und höherer Ordnung vernachlässigen.

Mit der Methode der vollständigen Induktion lässt sich dann zeigen, dass:

$$\bar{\Phi}_n(x-a, y-b) = \Phi_n(x-a, y-b)\left\{1-\delta\sum_{k=1}^{n}\Theta({}_n\gamma_k(x))\right\} \quad \text{.....(69),}$$

wobei
$$\Theta({}_n\gamma_k(x)) = \int^{{}_n\gamma_k(x)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \quad \text{.....(70),}$$

und
$$\left.\begin{aligned} {}_n\gamma_k(x) &= -\frac{kx+(n-k)a}{\sigma\sqrt{\tfrac{1}{2}nk(n-k)}} && \text{für } k\neq n \\ {}_n\gamma_n(x) &= -Mx, && \text{mit } M\to+\infty \end{aligned}\right\} \quad \text{.....(71).}$$

Setzt man $n=2$, so sieht man, dass (69) übergeht in (66 *a*), abgesehen von dem Glied mit δ^2. Weiter ist:

$$\begin{aligned}\bar{\Phi}_{n+1}(x-a, y-b) &= \frac{1}{N}\iint\bar{\Phi}_n(x'-a, y'-b)\,\Phi_1(x-x', y-y')\,dx'\,dy' \\ &\qquad\times\{1-\delta\Theta({}_{n+1}\gamma_{n+1}(x))\} \\ &= \frac{1}{N}\iint\Phi_n(x'-a, y'-b)\,\Phi_1(x-x', y-y') \\ &\qquad\times\left\{1-\delta\sum_{k=1}^{n}\Theta({}_n\gamma_k(x'))\right\}dx'\,dy'\,\{1-\delta\Theta({}_{n+1}\gamma_{n+1}(x))\},\end{aligned}$$

das ist unter Berücksichtigung der Formel (67) bei Integration der einzelnen Summanden:

$$= \Phi_{n+1}(x-a, y-b)\left\{1-\delta\sum_{k=1}^{n+1}\Theta({}_{n+1}\gamma_k(x))\right\},$$

wobei die Grössen in der Klammer $\{\ldots\}$ die in (70), (71) angegebene Bedeutung haben. Also gibt (69) die Verteilung nach dem n-ten Flug unter den oben erwähnten Bedingungen an. Allerdings hat diese Formel nur Gültigkeit wenn $\delta \ll 1$, sodass die Glieder mit δ^2, δ^3, ..., etc. wegen immer vorhandner Ungenauigkeit des σ^2 nicht ins Gewicht fallen.

(*b*) Es sei jetzt die Halbebene rechts der y-Achse mit einer Dichte N besetzt, während links der y-Achse keine Mücken vorhanden seien. Dann wird nach n Flügen bei der vorhergehenden Annahme über die Vernichtungsintensität die Dichte an einem Punkt (x, y) sein:

$$\bar{F}_n(x) = \iint_0 \bar{\Phi}_n(x-x', y-y')\,dx'\,dy' \quad \text{.....(72).}$$

Ich setze für $\bar{\Phi}_n(x-x', y-y')$

$$\Phi_n(x-x', y-y')\left\{1-\delta\sum_{k=1}^{n-1}\Theta({}_n\gamma_k(x, x'))\right\} \quad \begin{matrix}(1-\delta) & \text{für } x<0 \\ 1 & \text{„ } x>0\end{matrix} \quad \text{.....(73)}$$

ein, das stimmt nach (69) bis auf das quadratische Glied, das vernachlässigt werden kann.

Θ und ${}_n\gamma_k$ sind durch (70), (71) definiert mit x' statt dem dortigen a.

Es wird also:

$$\bar{F}_n(x) = \left\{ \iint_0 \Phi_n(x-x', y-y') \left[1 - \delta \sum_{k=1}^{n-1} \Theta({}_n\gamma_k) \right] dx'\, dy' \right\} \begin{matrix} (1-\delta) \text{ für } x<0, \\ 1 \quad \text{„} \quad x>0, \end{matrix}$$

nach (42)

$$= \left\{ F_n(-x) - \delta \sum_{k=1}^{n-1} \int_0 \left[\frac{1}{\sqrt{\pi n}\,\sigma} e^{-\frac{(x-x')^2}{n\sigma^2}} \int^{{}_n\gamma_x(x,x')} \frac{1}{\sqrt{2\pi}} e^{-\frac{x''^2}{2}} dx'' \right] dx' \right\} \begin{matrix} (1-\delta) \\ 1 \end{matrix}$$

$$= \left\{ F_n(-x) - \delta \sum_{k=1}^{n-1} \left[P_k(o) + \frac{P_k'(o)}{1!} x + \frac{P_k''(o)}{2!} x^2 + \dots \right] \right\} \begin{matrix} (1-\delta) \text{ für } x<0 \\ 1 \quad \text{„} \quad x>0 \end{matrix}$$

.....(74),

dabei ist:

$$P_k(x) = \int_0 \left\{ \frac{1}{\sqrt{\pi n}\,\sigma} e^{-\frac{(x-x')^2}{n\sigma^2}} \int^{{}_n\gamma_k(x,x')} \frac{1}{\sqrt{2\pi}} e^{-\frac{x''^2}{2}} dx'' \right\} dx' \quad \dots\dots\dots\dots\dots(75),$$

$$P_k'(x) = \frac{1}{\sqrt{\pi n}\,\sigma} e^{-\frac{x^2}{n\sigma^2}} \int^{K(x)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x''^2}{2}} dx'' - \frac{1}{\sigma\sqrt{2(n-k)}} \phi\left(\frac{x}{\sigma} \sqrt{\frac{2}{n-k}} \right) \dots(76),$$

wobei $$K(x) = -\frac{x}{\sigma}\sqrt{\frac{2k}{n(n-k)}}; \quad \phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \quad \dots\dots\dots\dots\dots(77).$$

Ich setze dann: $$P'(x) = A(x) - B(x),$$

also $$P^{(s)}(x) = A^{(s-1)}(x) - B^{(s-1)}(x) \quad \dots\dots\dots\dots\dots\dots\dots\dots(78).$$

Die $A^{(s-1)}(x)$ bzw. $B^{(s-1)}(x)$ entstehen durch Differentiation der einzelnen Summanden, aus denen $P'(x)$ besteht. Es wird also:

$$A'(x) = -\frac{2x}{\sigma^3\sqrt{\pi n^3}} e^{-\frac{x^2}{n\sigma^2}} \int^{K(x)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x''^2}{2}} dx' - \frac{1}{\sqrt{\pi n}\,\sigma^2} \sqrt{\frac{2k}{n-k}}\, \phi\left(\frac{x}{\sigma}\sqrt{\frac{2}{n-k}} \right)$$

.....(79),

und allgemein lässt sich dann durch vollständige Induktion zeigen, dass für $s \geqq 2$:

$$A^{(s-1)}(x) = -\frac{1}{n\sigma^2} \times \left\{ 2x A^{(s-2)}(x) + 2(s-2) A^{(s-3)}(x) + \frac{\sqrt{k}}{\sigma^{s-2}\sqrt{\pi}} \sqrt{\left(\frac{2}{n-k}\right)^{s-1}} \phi^{(s-2)}\left(\frac{x}{\sigma}\sqrt{\frac{2}{n-k}} \right) \right\}$$

.....(80),

und ebenso $$B^{(s-1)}(x) = \frac{1}{2\sigma^s} \sqrt{\left(\frac{2}{n-k}\right)^s} \phi^{(s-1)}\left(\frac{x}{\sigma}\sqrt{\frac{n}{n-k}} \right),$$

was man durch einfaches Differentiieren verifizieren kann.

Die Koeffizienten der Potenzreihe in (74) erhält man dann, indem man in (75) bis (80) $x = 0$ setzt. Die Ableitungen von $P_k(o)$ ergeben sich dabei ohne weiteres,

während $P_k(o)$ aus (75) noch durch eine kleine Rechnung vereinfacht werden muss. Es ist nämlich, wenn ich setze:

$$\frac{x'^2}{n\sigma^2}=\frac{\zeta^2}{2} \text{ und } \sqrt{\frac{n-k}{n}}=\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(81),$$

$$P_k(o)=\int_0\left\{\frac{1}{\sqrt{2\pi}}e^{-\frac{\zeta^2}{2}}\int^{-\zeta\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\right\}d\zeta=P(\alpha) \quad \ldots\ldots\ldots(75a).$$

Ich bilde dann:

$$\frac{dP(\alpha)}{d\alpha}=\int_0\frac{1}{\sqrt{2\pi}}e^{-\frac{\zeta^2}{2}}\left\{-\zeta\frac{1}{\sqrt{2\pi}}e^{-\frac{\alpha^2\zeta^2}{2}}\right\}d\zeta=\int_0\frac{-1}{2\pi}e^{-\frac{\zeta^2(1+\alpha^2)}{2}}\zeta d\zeta$$

$$=\frac{1}{2\pi(1+\alpha^2)}e^{-\frac{\zeta^2(1+\alpha^2)}{2}}\Big|_0^\infty=-\frac{1}{2\pi(1+\alpha^2)},$$

also $$P(\alpha)=\frac{1}{2\pi}\int^*\frac{-1}{1+\alpha^2}d\alpha=\frac{1}{2\pi}\operatorname{arc\,cotg}\alpha+C.$$

Nun ist aber nach (75a) $P(\alpha)$ für $\alpha=0$ gleich $\frac{1}{4}$. Also muss $C=0$ sein, wenn ich den arc cotg auf die Periode 0 bis π beschränke. Setze ich dann den Wert für α (nach (81)) ein, so wird:

$$P_k(o)=\frac{1}{2\pi}\operatorname{arc\,cotg}\sqrt{\frac{n-k}{k}} \qquad \text{(Periode 0 bis } \pi\text{)},$$

weiter nach (76): $$P'(o)=-\frac{1}{2\sigma\sqrt{\pi}}\left\{\frac{1}{\sqrt{n-k}}-\frac{1}{\sqrt{n}}\right\}.$$

Allgemein erhält man für die Nullstellen der Ableitungen mit ungeradem Index:

$$P_k^{(2s+1)}(o)=\frac{(-2)^{s-1}1.3.5.7\ldots(2s-1)}{\sqrt{\pi}\,\sigma^{2s+1}}\left\{\frac{1}{(n-k)^s\sqrt{n-k}}-\frac{1}{n^s\sqrt{n}}\right\}.$$

Für die Nullstellen der Ableitungen mit geradem Index ist mir keine allgemeine Darstellung geglückt. Es ist jedoch:

$$P_k''(o)=\frac{-1}{\pi n\sigma^3}\sqrt{\frac{k}{n-k}},$$

$$P_k^{iv}(o)=\frac{2(3n-2k)}{\pi n^2(n-k)\sigma^4}\sqrt{\frac{k}{n-k}},$$

$$P_k^{vi}(o)=\frac{-4(15n^2-20nk+8k^2)}{\pi n^3(n-k)^2\sigma^6}\sqrt{\frac{k}{n-k}},$$

$$P_k^{viii}(o)=\frac{24(35n^3-70n^2k+56nk^2-16k^3)}{\pi n^4(n-k)^3\sigma^8}\sqrt{\frac{k}{n-k}},$$

$$P_k^{x}(o)=\frac{-48(315n^4-840n^3k+1008n^2k^2-576nk^3+128k^4)}{\pi n^5(n-k)^4\sigma^{10}}\sqrt{\frac{k}{n-k}}.$$

* Unbestimmt integriert.

Die praktische Auswertung dieser Resultate wird vor allem von der Kenntnis der Grösse σ abhängen. Um dies σ zu bestimmen, wird man wohl von N Mücken einer bestimmten Art A experimentell eine Verteilung nach einem oder mehreren Flügen, ausgehend von einem Zentrum Q, beobachten müssen entsprechend Formel (33). Nachdem dann einmal von einer Mückenart A mit einer Flugzeit t_1 und einer linearen Fluggeschwindigkeit v_1 die Streuung des Abstandes von einem Punkt bekannt ist, kann man mit einiger Berechtigung die Streuung einer anderen Mückenart B mit einer Flugzeit t_2 und einer linearen Fluggeschwindigkeit v_2 berechnen, wie auf Seite 204—205 näher ausgeführt wurde.

Im nächsten Paragraphen soll nun an Hand einiger Beispiele die ungefähre Grösse des σ der *Anopheles quadrimaculatus* bestimmt werden.

7. *Bestimmung der "Streuung" für* Anopheles quadrimaculatus *nach Beobachtungen von M. H. Barber und T. B. Hayne.*

Es muss hier allerdings von vornherein betont werden, dass es sich bei den folgenden Berechnungen nur um rohe Näherungswerte handeln kann, da erstens die Zahl der zur Verfügung stehenden Beobachtungen zu klein ist und zweitens die auszuwertenden Versuche nicht eigentlich zu dem hier erstrebten Zweck angestellt wurden. Prinzipiell hätte man an verschiedenen Punkten und Tagen (wenn pro Tag ein Flug stattfindet) die Dichte der Mücken zu beobachten, wenn am Anfang des ersten Tages an einer bestimmten Stelle eine möglichst grosse Zahl gefärbter Mücken in Freiheit gesetzt wurden.

Aus den so gefundenen Daten wäre dann mit Hilfe der Ausgleichsrechnung aus Formel (33) das σ zu berechnen.

In der Praxis wird diese Bestimmung der Dichte an den verschiedenen Punkten nun ziemlich schwierig sein, da man im allgemeinen wohl kaum eine bestimmte Fläche ganz genau absuchen kann und da wegen Häufungen der Mückenanzahl in Stallungen, Wohnungen usw. zur Berechnung der Dichte an einem Punkt eine etwas andere als die abgesuchte Fläche anzusetzen ist.

Ich führe deshalb als neue unbekannte Grösse die Fläche F ein. Unter der Voraussetzung, dass F verhältnismässig klein gegen σ^2 ist, ist dann nach (33) die in der Entfernung r nach dem n-ten Flug auf F befindlich Anzahl Mücken:

$$Z_n(r) = (1-\delta)^n F\phi_n(r) = \frac{(1-\delta)^n FN}{\pi n\sigma^2}\, e^{-\frac{r^2}{n\sigma^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (82).$$

Es sei also der Vernichtungsfaktor δ örtlich und zeitlich konstant. Er gibt den nach einem Fluge ausgeschiedenen Bruchteil von der am Anfang des Fluges vorhanden gewesenen Gesamtzahl der Mücken an.

Bei den Versuchen hat man also an möglichst vielen Stellen, möglichst gleich grosse Flächen F gleich gut abzusuchen (in dem F ist mit berücksichtigt, dass von den wirklich vorhandenen Mücken nur ein Teil gefunden wird; insofern ist also F praktisch nie die wirklich abgesuchte Fläche). Die Unbekannten F, σ^2 und eventuell

δ sind dann so zu bestimmen, dass sich die Zahlen der an den einzelnen Stellen tatsächlich gefundenen Mücken möglichst gut den $Z_n(r)$ der Formel (82) anpassen. Es ist nun üblich, die Unbekannten so zu bestimmen, dass nicht die beobachteten Zahlen "$B_n(r)$" durch $Z_n(r)$, sondern dass $\log B_n(r)$ durch $\log Z_n(r)$ möglichst gut approximiert wird, da dadurch ein bedeutend gleichmässigerer Einfluss aller beobachteten Daten erreicht wird. Nach (82) ist nun:

$$\log Z_n(r) = \log \frac{N}{n\pi} + n \log(1-\delta) + \log \frac{F}{\sigma^2} - \frac{r^2 \log e}{n} \frac{1}{\sigma^2} \quad \ldots\ldots\ldots(83).$$

Ich setze dann:

$$\left.\begin{aligned} \log(1-\delta) &= x \\ \log \frac{F}{\sigma^2} &= y \\ \frac{1}{\sigma^2} &= z \end{aligned}\right\} \quad \ldots\ldots\ldots(84),$$

und suche nach der von Gauss herrührenden Methode der kleinsten Quadrate (s. Lit.-Verz. Nr. (10) bis (14)), die x, y, z so zu bestimmen, dass

$$\Sigma (\log B_n(r) - \log Z_n(r))^2$$

ein Minimum wird.

Von Barber und Hayne sind nun zwei Versuche ausgeführt worden, die ungefähr den hier verlangten Bedingungen entsprechen (s. Lit.-Verz. Nr. (15)).

TABELLE I.

n \ r	0	14·4	15
1	50 (1·699)	1 (0)	1 (0)
2	24 (1·380)	1 (0)	1 (0)
3	10 (1)	1 (0)	1 (0)
4	24 (1·380)	1 (0)	1 (0)
5	4 (0·602)	—	—

Die Zahlen in den Klammern sind die Briggs'schen Logarithmen der darüber stehenden Zahlen. r ist in 100 Fuss (= 30·48 m.) gemessen.

Tabelle I gibt von 2500 gefärbten ausgesetzten Mücken die Verteilung der in verschiedenen Entfernungen und an verschiedenen Tagen Wiedergefundenen an. Die Zahlen am Kopf der Tabelle sind die Entfernungen in 100 Fuss, während links die Anzahl der seit der Aussetzung verstrichenen Tage steht.

Setzt man nun für die x, y, z vom (84) in Formel (83) die Näherungswerte:

$$x_0 = 0, \quad y_0 = -2, \quad z_0 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (85),$$

so erhält man 13 sogenannte Fehlergleichungen:

$$\left.\begin{aligned} v_\nu &= n\xi + 1\eta - \frac{r^2 \log e}{n}\zeta - \left(\log B_n(r) - \log\frac{N}{n\pi} + 2\right) \\ &= a_\nu\xi + b_\nu\eta + \quad c_\nu\zeta - \quad l_\nu \end{aligned}\right\} \ldots\ldots\ldots (86),$$

in die die aus Tabelle I zu entnehmenden Werte einzusetzen sind, also:

$$\begin{aligned} v_1 &= 1\xi + 1\eta - \ 0 - 0{\cdot}798, \\ v_2 &= 2\xi + 1\eta - \ 0 - 0{\cdot}780, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ v_6 &= 1\xi + 1\eta - 90 + 0{\cdot}901, \\ v_7 &= 2\xi + 1\eta - 45 + 0{\cdot}600, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \text{ usw.} \end{aligned}$$

Die ξ, η, ζ, das sind die an x, y, z anzubringenden Verbesserungen, sind nun so zu bestimmen, dass $[vv]$* ein Minimum wird.

Dazu bildet man die Summen der Produkte der in den Fehlergleichungen stehenden Koeffizienten a_ν, b_ν, c_ν und l_ν, aus denen man dann die sogenannten Normalgleichungen erhält:

$$\left.\begin{aligned} [aa]\xi + [ab]\eta + [ac]\zeta &= [al] \\ [ba]\xi + [bb]\eta + [bc]\zeta &= [bl] \\ [ca]\xi + [cb]\eta + [cc]\zeta &= [cl] \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (87).$$

Die Lösung dieses Gleichungssystems gibt die zu den Näherungswerten hinzuzufügenden Verbesserungen an.

Die Normalgleichungen unseres Beispiels lauten:

$$\begin{aligned} 115\ \xi + \ 35\ \eta - \ 750{\cdot}4\,\zeta &= \ 1{\cdot}272, \\ 35\ \xi + \ 13\ \eta - \ 390{\cdot}8\,\zeta &= -0{\cdot}813, \\ -750{\cdot}4\,\xi - 390{\cdot}8\,\eta + 25070\ \zeta &= 265{\cdot}9. \end{aligned}$$

Die Lösungen sind:

$$\xi = -0{\cdot}123; \quad \eta = 0{\cdot}89; \quad \zeta = 0{\cdot}021.$$

Also wird nach (85):

$$x = -0{\cdot}123; \quad y = -1{\cdot}11; \quad z = 0{\cdot}021.$$

Für die mittleren Fehler ergeben sich die Werte:

$$m_x = \pm\,0{\cdot}088; \quad m_y = \pm\,0{\cdot}32; \quad m_z = \pm\,0{\cdot}0035.$$

* Es bedeutet: $[aa] = \sum_\nu a_\nu a_\nu$, $[ab] = \sum_\nu a_\nu b_\nu$, usw.

Einzelheiten bezüglich der Ableitung der Theorie der Methode der kleinsten Quadrate sind u.a. in den auf Seite 226 angegebenen Werken zu finden.

Nach (84) erhält man dann die gesuchten Grössen:

$$1-\delta = 0{\cdot}75 \pm 0{\cdot}15^{*}; \quad \frac{F}{\sigma^2} = 0{\cdot}078 \pm 0{\cdot}057;$$

$$\sigma = (6{\cdot}9 \pm 0{\cdot}6)\,100 \text{ Fuss} = (210 \pm 20) \text{ m}.$$

Wie nach der Ausführung des Versuches zu erwarten, ist die Ungenauigkeit von F am grössten.

Der Wert für $1-\delta$ stimmt ausserordentlich gut mit auf ganz andere Weise angestellten Berechnungen von Herrn Prof. Martini überein (s. Lit.-Verz. Nr. (16)).

Aus σ folgt nach (31):

$$l = \frac{\sigma}{2}\sqrt{\pi} = 186 \text{ m}.$$

Prof. Pearson (s. Lit.-Verz. Nr. (3), S. 36) schätzte nun die Grösse von l zwischen 100 und 500 Yard (91·4 bis 457·2 m.) in guter Übereinstimmung mit dem hier gefundenen Wert.

TABELLE II.

n \ r	0		14·4	15
	Beob.	Theor.		
1	50	46·5	1 0·6	1 0·4
2	24	17·5	1 2·0	1 1·6
3	10	8·8	1 2·1	1 1·8
4	(24	4·9)	1 1·7	1 1·5
5	4	2·9	—	—

Tabelle II gibt eine Gegenüberstellung der beobachteten (links in den Spalten) mit den aus (82) nach Einsetzung der oben gefundenen Zahlen berechneten Werten.

Ein Mass für die Güte der Anpassung ist die von Pearson stammende χ^2 Methode (s. Lit.-Verz. Nr. (18), S. 157—75, und Nr. (9), S. xxxi). Man hat zu diesem Zweck zu bilden:

$$\chi^2 = \Sigma \frac{(\text{beob. Wert} - \text{theor. Wert})^2}{\text{theor. Wert}}.$$

* Man erhält die mittleren Fehler aus denen für x, y, z mit Hilfe des Gauss'schen Fehlerfortpflanzungsgesetzes:

$$m_f = \pm\sqrt{\left(m_x \frac{\partial f}{\partial x}\right)^2 + \left(m_y \frac{\partial f}{\partial y}\right)^2 + \ldots}.$$

m_f ist der mittlere Fehler einer Funktion $f(x, y, \ldots)$, wenn m_x, m_y, ... die voneinander unabhängigen mittleren Fehler von x, y, ... sind. Also z.B. wenn

$$f(x, y, \ldots) = 1 - \delta = 10^x$$

ist, wird:

$$m_f = \pm 0{\cdot}088 \frac{\log_e 10}{10^{0{\cdot}125}} = \pm 0{\cdot}15.$$

(Die Zahlenwerte sind aus obigem Versuch genommen.)

Für jede betrachtete Anzahl von Klassen entspricht dann den verschiedenen Werten von χ^2 ein P, d.i. die Wahrscheinlichkeit dafür, dass eine durch Zufall entstehende Abweichung der beobachteten von der theoretischen Verteilung ebenso gross oder grösser ist als die der untersuchten Verteilung. Die zu den verschiedenen χ^2 gehörigen Werte P sind tabelliert (s. Lit.-Verz. Nr. (9), S. 26, und Nr. (19)).

In unserem Beispiel erhält man bei ausser acht lassen des ganz aus der Reihe fallenden Wertes für $n=4$, $r=0$ für χ^2 den Wert 6·54, dem entspricht:

$$P = 0{\cdot}8.$$

Die Anpassung ist also als überaus gut zu bezeichnen. Man nennt nämlich im allgemeinen eine Anpassung noch gut, solange $p > 0{\cdot}1$ (näheres hierüber siehe Lit.-Verz. Nr. (20), S. 104—8).

Der zweite Versuch von Barber und Hayne.

TABELLE III.

n \ r	0	1	2	5	14·4
1	277 (2·442)	1 (0)	1 (0)	1 (0)	1 (0)
2	29 (1·462)	4 (0·602)	2 (0·301)	—	—
3	4 (0·602)	—	—	—	—
4	2 (0·301	1 (0)	—	—	—
6	1 (0)	—	—	—	—

Tabelle III gibt diesmal die Verteilung von 2400 Mücken. Sonst ist die Bedeutung der Zahlen die gleiche wie die der Tabelle I des ersten Versuches (S. 220). Wie man ohne weiteres sieht, sind die wiedergefundenen Mücken diesmal bedeutend regelloser verteilt, und es ist anzunehmen, dass durch das Absuchen oder sonstwie systematische Fehler in bezug auf die Anzahl der an den einzelnen Stellen wirklich vorhanden gewesenen Mücken vorgekommen sind. Behandelt man nun die Daten dieses zweiten Versuches genau wie die des ersten, so erhält man:

$$1-\delta = 1{\cdot}024; \quad \frac{F}{\sigma^2} = 0{\cdot}009; \quad \sigma = 9{\cdot}8\ 100 \text{ Fuss}.$$

Der Wert für $1-\delta$ ist jedoch unmöglich, denn auf jeden Fall muss $1-\delta < 1$ sein. Ich habe deshalb bei der Auswertung des zweiten Versuches die Grösse $1-\delta$ als bekannt gleich 0·7 angesetzt, welchen Wert man auf Grund des vorigen Versuches und der erwähnten Berechnungen von Prof. Martini wohl als einigermassen richtig ansehen kann.

Setzt man also in (82) $1-\delta = 0{\cdot}70$,
und führt im übrigen die analoge Rechnung wie beim ersten Versuch mit den entsprechenden Zahlen der Tabelle III durch, so erhält man als Normalgleichungen:

$$12\ \eta - 104{\cdot}2\zeta = 2{\cdot}812,$$
$$104{\cdot}2\eta - 8222\ \zeta = 74{\cdot}8,$$

mit den Lösungen: $\eta = 0{\cdot}35$; $\zeta = 0{\cdot}0135$.

Also wegen (85): $y = -1{\cdot}65$; $z = 0{\cdot}0135$.

Die ebenfalls aus den Koeffizienten der Normalgleichungen zu berechnenden mittleren Fehler sind:

$$m_y = \pm 0{\cdot}24; \quad m_z = \pm 0{\cdot}0092.$$

Wegen (84) wird dann endlich:

$$\frac{F}{\sigma^2} = 0{\cdot}0224 \pm 0{\cdot}0124; \quad \sigma = (8{\cdot}6 \pm 2{\cdot}9)\,100\ \text{Fuss} = (260 \pm 90)\ \text{m}.$$

Wie man sieht ist mit den Resultaten des ersten Versuches eine weitgehende Uebereinstimmung innerhalb des Bereiches der mittleren Fehler vorhanden.

TABELLE IV.

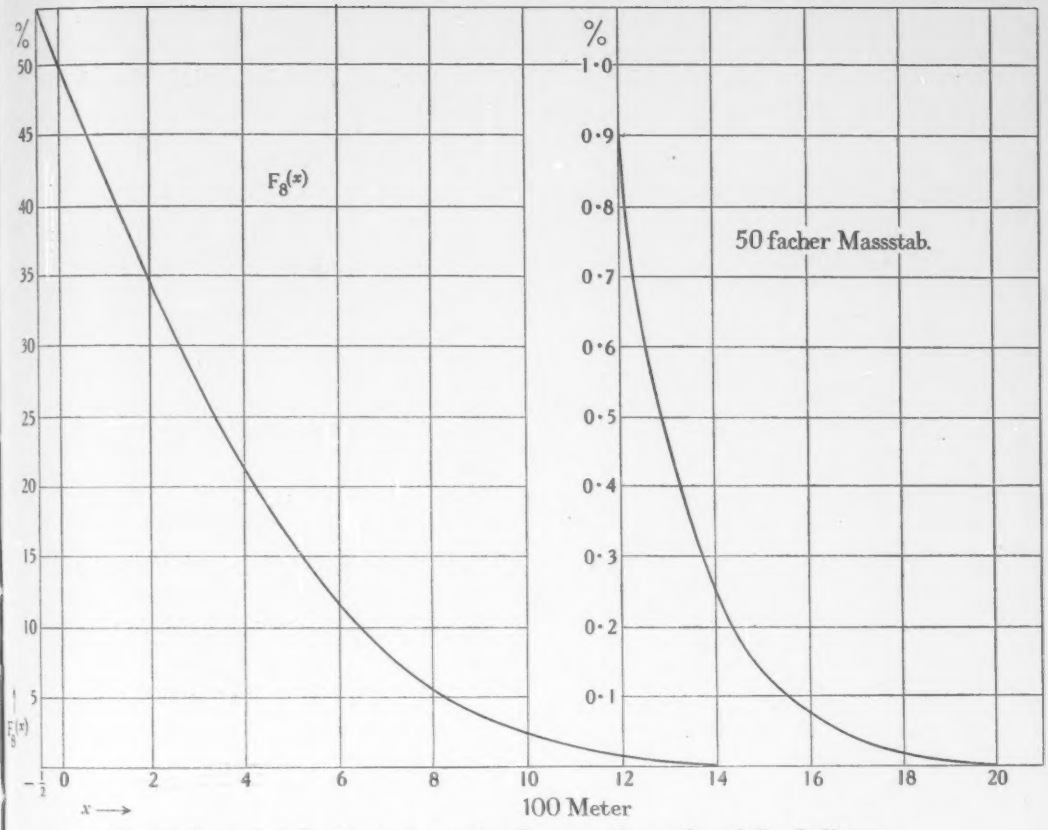
n \ r	0		1		2		5		14·4	
1	277	12·0	1	11·8	1	11·3	1	8·5	1	0·75
2	29	4·2	4	4·2	2	4·1	—		—	
3	4	2·0	—		—		—		—	
4	2	1·0	1	1·9	—		—		—	
6	1	0·3	—		—		—		—	

Diese Tabelle gibt wieder eine Gegenüberstellung der theoretischen (rechts in den Spalten) mit den beobachteten (links in den Spalten) Zahlen. Man sieht sofort, dass im ganzen genommen eine sehr schlechte Anpassung der theoretischen an die beobachteten Daten vorliegt. Ein Vergleich mit Hilfe der erwähnten χ^2 Methode ergibt für P einen weit kleineren Wert als 0·0001. Die 5 stärksten Abweichungen sind nun prozentual im Verhältnis zu den übrigen derartig gross, dass für sie ein systematischer Fehler anzunehmen ist. Man ist deshalb in gewissem Sinne berechtigt, diese bei Beurteilung der Anpassungsgüte auszuschalten. Für die dann übrigbleibenden 7 Beobachtungen erhält man:

$$\chi^2 = 6{\cdot}2 \quad \text{das gibt:}\ P = 0{\cdot}4,$$

was immerhin noch sehr gut genannt werden muss.

Man kann also mit einigem Recht auf Grund dieser beiden Versuche für σ einen Wert von etwa 250 m. ansetzen und damit die in dieser Arbeit behandelten Verteilungen für die *Anopheles quadrimaculatus* zahlenmässig berechnen. In folgender Skizze habe ich z.B. die Verteilung von einer Halbebene mit der Dichte 100 in die andere nach 8 Flügen dargestellt.



Die Abzissen sind die Abstände von der Grenzgeraden, während die Ordinaten die in den verschiedenen Entfernungen vorhandenen Dichten, d.h. Anzahl pro Flächeneinheit, angeben. Die Längeneinheit sei 100 m. (Flächeneinheit also 10,000 m.²). Vermehrung und Absterben sollen sich aufheben, also $\delta = 0$.

Dann wird die Dichte im Abstand $x \times 100$ m. von der Grenzlinie nach 8 Flügen zufolge (42):

$$F_8(x) = 1 - \frac{1}{\sqrt{\pi n}\,\sigma} \int^{x} e^{-\frac{x'^2}{n\sigma^2}}\, dx' \qquad (\text{mit } n = 8,\ \sigma = 2{\cdot}5)$$

$$= 1 - \frac{1}{\sqrt{2\pi}} \int^{x/5} e^{-\frac{x'^2}{2}}\, dx'.$$

Die Werte dieses Integrals habe ich aus den Pearson'schen Tabellen entnommen (s. Lit.-Verz. Nr. (9), S. 2 bis 9).

$\frac{1}{\sqrt{2\pi}} \int^{x} e^{-\frac{x'^2}{2}}\, dx'$ ist dort mit $\frac{1}{2}(1-\alpha)$ bezeichnet.

Der zweite Kurvenzug ist die Verlängerung des ersten bei 50-fach vergrössertem Massstab.

Herrn Prof. Dr P. Riebesell sowie Herrn Prof. Dr E. Martini möchte ich auch noch an dieser Stelle meinen besonderen Dank aussprechen für das rege Interesse, dass sie an der Fertigstellung dieser Arbeit stets bewiesen haben.

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THOUGHTS SUGGESTED BY THE PAPERS OF MESSRS WELCH AND KOŁODZIEJCZYK (*Biometrika*, Vol. XXVII. pp. 145—190).

By KARL PEARSON.

These authors take populations $\Pi_1, \Pi_2 \ldots \Pi_i \ldots \Pi_k$ and suppose that in any one of these k populations, the distribution of the y's for a given x is normal and homoscedastic. Accordingly the frequency surfaces must be of the form

$$z_i = \phi_i(x)\, e^{-\frac{1}{2\bar{\sigma}^2_{a,i}}(y-\bar{\alpha}_i-\bar{\beta}_i x)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

where $\bar{\sigma}_{a,i}$ is the array standard deviation, and $\bar{\alpha}_i$ and $\bar{\beta}_i$ the constants in the population Π_i, and nothing is stated about the form of $\phi_i(x)$, which *à priori* may be considered continuous or discontinuous. For the truth of the following results it does not matter whether we take a summation or integrate with regard to x between arbitrary limits. But the reader must bear in mind that we are summing for the parent population and *not* for any sample taken from it. We shall use $m_{1,i}$ and $m_{2,i}$ for the x and y means of the population, and $\bar{\sigma}_{1,i}$ and $\bar{\sigma}_{2,i}$ for their standard deviations, $\bar{\rho}_i$ will be their correlation coefficient. For a sample from this population, we shall use $\bar{x}_i$, $\bar{y}_i$ for the means $\sigma_{1,i}$, $\sigma_{2,i}$ for the standard deviations and r_i for the correlation coefficient, and $\sigma_{a,i}$ for an array standard deviation of the y's for a given x_i.

If $\tilde{y}_{x,i}$ be the mean of the x array of y's the form of (i) shows us at once that

$$\tilde{y}_{x,i} = \bar{\alpha}_i + \bar{\beta}_i x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

and there is accordingly a straight regression line for y on x.

Summing for all values of x, it follows whatever be the form of $\phi_i(x)$ that

$$m_{2,i} = \bar{\alpha}_i + \bar{\beta}_i m_{1,i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii}^{bis}\text{)}.$$

We will now show that for any surface given by (i)

$$\bar{\beta}_i = \bar{\rho}_i \bar{\sigma}_{2,i}/\bar{\sigma}_{1,i}.$$

We have, if N_i be the size of the population Π_i,

$$\begin{aligned}
S_x x\phi_i(x) \int_{-\infty}^{+\infty} e^{-\frac{1}{2\bar{\sigma}^2_{a,i}}(y-\bar{\alpha}-\bar{\beta}_i x)^2}\, y\,dy &= N_i\,(\bar{\rho}_i \bar{\sigma}_{1,i} \bar{\sigma}_{2,i} + m_{2,i} m_{1,i}) \\
&= S_x x\phi_i(x) \int_{-\infty}^{+\infty} (y - \bar{\alpha}_i - \bar{\beta}_i x + \bar{\alpha}_i + \bar{\beta}_i x)\, e^{-\frac{1}{2\bar{\sigma}^2_{a,i}}(y-\bar{\alpha}_i-\bar{\beta}_i x)^2}\, dy \\
&= S_x x\phi_i(x)\,[0 + \bar{\alpha}_i + \bar{\beta}_i x]\sqrt{2\pi}\,\bar{\sigma}_{a,i} \\
&= [\bar{\alpha}_i m_{1,i} + \bar{\beta}_i (\bar{\sigma}^2_{1,i} + m^2_{1,i})]\, S_x \phi_i(x) \sqrt{2\pi}\,\bar{\sigma}_{a,i}.
\end{aligned}$$

But $S_x \phi_i(x) \sqrt{2\pi}\, \bar{\sigma}_{a,i} = N_i$ and therefore by (ii)

$$N_i(\bar{\rho}_i \bar{\sigma}_{1,i} \bar{\sigma}_{2,i} + m_{1,i} m_{2,i}) = N_i(\bar{\beta}_i \bar{\sigma}^2_{1,i} + m_{1,i} m_{2,i}),$$

or
$$\bar{\beta}_i = \bar{\rho}_i \bar{\sigma}_{2,i} / \sigma_{1,i} \quad \text{.....(iii)},$$

whatever be the function $\phi_i(x)$.

We may now proceed further and ask what is the form of $\sigma^2_{a,i}$ for a surface like (i)?

We will find $\sigma^2_{2,i}$.

$$N_i(\sigma^2_{2,i} + m^2_{2,i}) = S_x \phi_i(x) \sqrt{2\pi}\, \bar{\sigma}_{a,i} (\sigma^2_{2,i} + m^2_{2,i})$$

$$= S_x \phi_i(x) \int_{-\infty}^{+\infty} (y - \bar{\alpha}_i - \bar{\beta}_i x + \bar{\alpha}_i + \bar{\beta}_i x)^2 e^{-\frac{1}{2\bar{\sigma}_{a,i}} (y - \alpha_i - \beta_i x)^2} dy$$

$$= S_x \phi_2(x) [\bar{\sigma}^2_{a,i} \sqrt{2\pi}\, \bar{\sigma}_{a,i} + 0 + (\bar{\alpha}_i^2 + 2\bar{\alpha}_i \beta_i x + \bar{\beta}_i x^2) \sqrt{2\pi}\, \sigma_{a,i}].$$

Thus $N_i(\sigma^2_{2,i} + m^2_{2,i}) = N_i[\bar{\sigma}^2_{a,i} + \bar{\alpha}_i^2 + 2\bar{\alpha}_i \bar{\beta}_i m_{1,i} + \bar{\beta}_i^2 (\bar{\sigma}^2_{1,i} + m^2_{1,i})]$.

Hence by (ii) and (iii)

$$\sigma^2_{2,i} + m^2_{2,i} = \bar{\sigma}^2_{a,i} + (\bar{\alpha}_i + \bar{\beta}_i m_{1,i})^2 + \bar{\beta}_i^2 \bar{\sigma}^2_{1,i},$$

$$\sigma^2_{2,i} (1 - \bar{\rho}_i^2) = \bar{\sigma}^2_{a,i} \quad \text{.....(iv)},$$

which is exactly the form $\bar{\sigma}^2_{a,i}$ would take for a normal surface, and this whatever be the form of $\phi_i(x)$.

Now if an additional hypothesis be made that $\bar{\sigma}_{a,i}$ is to be the same for all the k populations Π_1, Π_2 ... Π_i ... Π_k, then for those populations we must have

$$\bar{\sigma}^2_{2,i} \propto \frac{1}{1 - \bar{\rho}_i^2} \quad \text{.....(v)},$$

which signifies that for these k populations the variance of y in each of them varies inversely as unity minus the square of its correlation coefficient. It would seem difficult to discover any series of populations for which such a relation would hold. Personally I am not able to think of any series of populations for which the variance and the correlation would obey such an artificial, not to say fantastic, relationship except:

(*a*) a series of populations with the same variance for y and the same correlation coefficient, i.e.

$$\bar{\sigma}_{2,i} = \bar{\sigma}'_{2,i'} \text{ and } \bar{\rho}_i = \bar{\rho}_{i'} \text{ for all values of } i \text{ and } i' \quad \text{.....(vi)},$$

or, (*b*) a series of populations taken from different parts of one and the same population. This would involve not only (vi), but $\bar{\alpha}_i$ and $\bar{\beta}_i$ being the same, to say nothing of $m_{1,i}$ and $m_{2,i}$, as well as

$$\bar{\sigma}_{1,i} = \bar{\sigma}_{1,i'} \text{ for all values of } i \text{ and } i' \quad \text{.....(vii)}.$$

Thus case (*b*) involves far more stringent hypotheses than (*a*).

But if we accept (*a*) and then superpose upon it the additional hypothesis that the regression coefficient of these populations, namely $\bar{\beta}_i$, is to be constant, we

have fixed $\bar{\rho}_i\bar{\sigma}_{2,i}/\bar{\sigma}_{1,i}$ and accordingly $\bar{\sigma}_{1,i}$ must be considered constant in this case. (a) therefore connotes that the three constants $\bar{\rho}_i$, $\bar{\sigma}_{1,i}$, $\bar{\sigma}_{2,i}$ are fixed for all k surfaces, but the means may be different, which is not possible in (b).

But we can now go further. Clearly $\bar{\sigma}_{1,i}^2(1-\bar{\rho}_i^2)$ will be constant for all k populations and accordingly the arrays of x on y will be homoscedastic. This does not refer to the samples taken from those populations which may not be random at all, but to the populations Π_1, $\Pi_2 \ldots \Pi_i \ldots \Pi_k$ themselves.

Now the most general equation to a surface having homoscedasticity both ways* is

$$z = z_0 e^{-\left(\frac{\gamma}{\lambda_1\lambda_2}e^{\lambda_1 x+\lambda_2 y}-\frac{\gamma}{\lambda_1^2 m_1}e^{\lambda_1 x}-\frac{\gamma}{\lambda_2^2 m_2}e^{\lambda_2 y}-c_1'' x-c_2'' z\right)} \quad \ldots\ldots\ldots\text{(viii)},$$

where γ, m_1, m_2, λ_1, λ_2, c_1'' and c_2'' are arbitrary constants. This has for its regression curves

$$\tilde{y}_x = c_2 - \frac{1}{\lambda_2}\log\left(1-\frac{\lambda_2}{\lambda_1}m_2 e^{\lambda_1 x}\right) \quad \ldots\ldots\ldots\text{(ix)}$$

and

$$\tilde{x}_y = c_1 - \frac{1}{\lambda_1}\log\left(1-\frac{\lambda_1}{\lambda_2}m_1 e^{\lambda_2 y}\right) \quad \ldots\ldots\ldots\text{(x)}.$$

Now by hypothesis (i) we are to have (ix) linear.

Let us expand the logarithm and the exponential. We have

$$\begin{aligned}
\tilde{y}_x &= c_2 + \frac{1}{\lambda_2}\left(\frac{\lambda_2}{\lambda_1}m_2 e^{\lambda_1 x}+\frac{\lambda_2^2 m_2^2}{2\lambda_1^2}e^{2\lambda_1 x}+\frac{\lambda_2^3 m_2^3}{3\lambda_1^3}e^{3\lambda_1 x}+\ldots\right)\\
&= c_2 + \frac{m_2}{\lambda_1}\left(1+\lambda_1 x+\frac{1}{2!}\lambda_1^2 x^2+\ldots\right)\\
&\quad + \frac{m_2^2\lambda_2}{2\lambda_1^2}\left(1+2\lambda_1 x+\frac{1}{2!}4\lambda_1^2 x^2+\ldots\right)\\
&\quad + \frac{m_2^3\lambda_2^2}{3\lambda_1^2}\left(1+3\lambda_1 x+\frac{1}{2!}9\lambda_1^2 x^2+\ldots\right)\\
&\quad + \text{etc.}\\
&= c_2 + m_2\left(\frac{1}{\lambda_1}\right)+\frac{m_2^2\lambda_2}{2\lambda_1^2}+\frac{m_2^3\lambda_2^2}{3\lambda_1^3}+\ldots\\
&\quad + x\left(m_2+m_2^2\frac{\lambda_2}{\lambda_1}+m_2^3\frac{\lambda_2^2}{\lambda_1^2}+\ldots\right)\\
&\quad + \frac{x^2}{2!}\left(m_2\lambda_1+2m_2^2\lambda_2+3m_2^3\frac{\lambda_2^2}{\lambda_1}+\ldots\right)\\
&\quad + \text{etc.}
\end{aligned}$$

To make the right-hand side linear we are bound to make λ_2/λ_1 finite, and both λ_1 and λ_2 zero. The infinite constant now may be added to the arbitrary constant c_2; the coefficients of all powers of x vanish except that of x which is finite and we have

$$\tilde{y}_x = c_2' + m_2' x \quad \ldots\ldots\ldots\text{(xi)}.$$

* S. Narumi, *Biometrika*, Vol. xv. pp. 82—85.

But the very same values of λ_1 and λ_2 turn (x) into

$$\tilde{x}_y = c_1' + m_1' y \quad \text{.....(xii).}$$

Accordingly we cannot have a frequency surface homoscedastic both ways and with linear regression one way, without linear regression the other way.

But a frequency surface linear both ways and homoscedastic both ways can only be a bivariate normal surface. If this result be not familiar, it can be deduced from (viii) by making the ratio λ_2/λ_1 finite and λ_1 and λ_2 both zero.

Thus it appears to me, without being dogmatic, that unless a series of surfaces can be produced in which the relation (v) holds other than those supplied by (*a*) or (*b*), we do not narrow our hypothesis by supposing that the populations Π_1, Π_2, ... Π_i ... Π_k are all normal surfaces differing the one from the other only in the position of their means. These means will not of necessity be collinear; this depends on the constancy of $\bar{\alpha}_i$, supposing that we have accepted the hypotheses that $\bar{\sigma}_{a,i}$ and $\bar{\beta}_i$ are the same for all the populations.

The thoughts here recorded do not question the validity of the processes by which the authors cited would deduce the constancy of $\bar{\sigma}_{a,i}$, $\bar{\alpha}_i$ or $\bar{\beta}_i$. But they do tend to suggest that the surface (i) from which they start is, when the hypotheses $\bar{\sigma}_{a,i} = \bar{\sigma}_{a,i'}$ and $\bar{\beta}_i = \bar{\beta}_{i'}$ are applied, not so general as they, perhaps, imagine and that accordingly there may be other, possibly as simple, methods of attacking their problem as that provided by them.

Let us consider first samples which may be supposed taken from a bivariate parent normal surface defined by means m_1, m_2, σ_1, σ_2, and correlation ρ, and let any sample be defined by $\bar{x}$, $\bar{y}$, Σ_1, Σ_2 and r. The frequency surface in five-way space is known to be*

$$z = z_0 e^{-\frac{1}{2}\left\{\frac{(\bar{x}-m_1)^2}{s_1^2} - \frac{2\rho(\bar{x}-m_1)(\bar{y}-m_2)}{s_1 s_2} + \frac{(\bar{y}-m_2)^2}{s_2^2}\right\}}$$
$$\times e^{-\frac{1}{2}\left\{\frac{\Sigma_1^2}{s_1^2} - \frac{2\rho r \Sigma_1 \Sigma_2}{s_1 s_2} + \frac{\Sigma_2^2}{s_2^2}\right\}} \left(\frac{\Sigma_1}{s_1}\right)^{M-2} \left(\frac{\Sigma_2}{s_2}\right)^{M-2} (1-r^2)^{\frac{M-4}{2}} [d\bar{x}\, d\bar{y}\, d\Sigma_1\, d\Sigma_2\, dr]$$
$$= z_0 F_1(\bar{x}, \bar{y}) \times F_2(\Sigma_1, \Sigma_2, r)\,[d\bar{x}_1 d\bar{y}_1 d\Sigma_1 d\Sigma_2 dr] \text{ say, for brevity .....(xiii).}$$

There are five variables here, i.e. those of the sample $\bar{x}, \bar{y}, \Sigma_1, \Sigma_2, r$. I propose to replace them by the five other variables $\bar{x}$, $\alpha = \bar{y} - \beta\bar{x}$, $\beta = r\Sigma_2/\Sigma_1$, Σ_1, and $\gamma = \Sigma_2\sqrt{1-r^2}$†. In doing this I shall throw any numerical factor or any function of the parental constants m_1, m_2, s_1, s_2 and ρ only, which arises as a multiplier of F_1 or F_2, into z_0.

I shall begin first by getting rid of $\bar{y} = \alpha + \beta\bar{x}$ in $F_1(\bar{x}, \bar{y})$. After re-arranging in powers of $\bar{x}$, we have

$$F_1(\bar{x}, \alpha, \beta) = e^{-\frac{1}{2}\left[Q\left\{\frac{\bar{x}-m_1}{s_1} - \left(\rho - \beta\frac{s_1}{s_2}\right)\frac{\alpha - m_2 + \beta m_1}{s_2 Q}\right\}^2 + \frac{(\alpha - m_2 + \beta m_1)^2(1-\rho^2)}{s_2^2 Q}\right]},$$

where
$$Q = 1 - \frac{2\rho\beta s_1}{s_2} + \frac{\beta^2 s_1^2}{s_2^2} = 1 - \frac{2\rho\beta\sigma_1}{\sigma_2} + \beta^2\frac{\sigma_1^2}{\sigma_2^2} \quad \text{.....(xiv).}$$

* We write $s_1^2 = \sigma_1^2(1-\rho^2)/M$, and $s_2^2 = \sigma_2^2(1-\rho^2)/M$, where M is the size of the sample.

† The reader must be careful to bear in mind that α, β, γ here are all sample and not parent-population values.

Now $\bar{x}$ does not occur in $F_2(\Sigma_1, \Sigma_2, r)$ and thus we see if α and β are fixed $\bar{x}$ still follows a normal curve round the value

$$\text{mean } \bar{x} = m_1 - \frac{s_1}{s_2}\left(\rho - \beta\frac{s_1}{s_2}\right)\frac{\alpha - m_2 + \beta m_1}{Q} \quad \text{.....(xv),}$$

with standard deviation $\sigma_{\bar{x}\cdot\alpha,\beta} = s_1/\sqrt{Q}$ .....(xvi).

We see that this standard deviation is independent of α, which only influences $\bar{x}$. We now transform our five variables $\bar{x}$, $\bar{y}$, Σ_1, Σ_2, r to the variables $\bar{x}$, α, Σ_1, β, and γ. Evaluating the determinant we find

$$d\bar{x}\,d\bar{y}\,d\Sigma_1\,d\Sigma_2\,dr = \frac{\Sigma_1}{\Sigma_2}\sqrt{1-r^2}\,d\bar{x}\,d\alpha\,d\Sigma_1\,d\beta\,d\gamma.$$

We can now integrate out for $\bar{x}$ and have a four-way frequency surface:

$$z' = z_0'\sqrt{Q}\,e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2(1-\rho^2)}{s_2^2 Q}}\frac{\Sigma_1}{\Sigma_2}\sqrt{1-r^2}F_2(\Sigma_1, \Sigma_2, r) \qquad [d\alpha\,d\Sigma_1\,d\beta\,d\gamma].$$

The next stage is to express the function $\dfrac{\Sigma_1\sqrt{1-r^2}}{\Sigma_2}F_2(\Sigma_1, \Sigma_2, r)$ in terms of Σ_1, β and γ. What we have to express is

$$e^{-\frac{1}{2}\left(\frac{\Sigma_1^2}{s_1^2} - \frac{2\rho r\Sigma_1\Sigma_2}{s_1 s_2} + \frac{\Sigma_2^2}{s_2^2}\right)}\left(\frac{\Sigma_1}{s_1}\right)^{M-1}\left(\frac{\Sigma_2}{s_2}\right)^{M-3}(1-r^2)^{\frac{M-3}{2}},$$

and this leads to

$$e^{-\frac{1}{2}\frac{\Sigma_1^2}{s_1^2}\left(1 - \frac{2\rho\beta s_1}{s_2} + \beta^2\frac{s_1^2}{s_2^2}\right) - \frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\Sigma_1}{s_2}\right)^{M-1}\left(\frac{\gamma}{s_2}\right)^{M-3},$$

or

$$e^{-\frac{1}{2}\frac{\Sigma_1^2}{s_1^2}Q}\left(\frac{\Sigma_1}{s_1}\right)^{M-1} e^{-\frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\gamma}{s_2}\right)^{M-3}.$$

We can now write our fourfold surface of frequency in the form

$$z'\,d\alpha\,d\Sigma_1\,d\beta\,d\gamma = z_0'\frac{1}{\sqrt{Q}}e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2(1-\rho^2)}{s_2^2 Q}}d\alpha\,d\beta$$
$$\times e^{-\frac{1}{2}\frac{\Sigma_1^2}{s_1^2}Q}\left(\frac{\Sigma_1}{s_1}\right)^{M-1}d\Sigma_1\,e^{-\frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\gamma}{s_2}\right)^{M-3}d\gamma \text{.....(xvii).}$$

This admits at once of being integrated out with regard to Σ_1, leading us to the complete Γ-function $\frac{1}{2}M$, and other factors which can be thrown into the z_0', as well as the important factor $\dfrac{1}{Q^{M/2}}$. Thus we reduce our expression to a three-way surface of frequency

$$z''\,d\alpha\,d\beta\,d\gamma = z_0''\frac{1}{Q^{\frac{1}{2}(M+1)}}e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2(1-\rho^2)}{s_2^2 Q}}d\alpha\,d\beta \times e^{-\frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\gamma}{s_2}\right)^{M-3}d\gamma \text{...(xviii).}$$

This result is of considerable importance; we see that the three-way frequency surface breaks up into two independent frequency distributions, the one the bivariate frequency surface of correlation for α and β, and the other the frequency

curve for the distribution of γ, the standard deviation of the arrays of y on x, i.e. $\Sigma_2\sqrt{1-r^2}$. This latter, as pointed out in my memoir of 1926*, is precisely of the same form as the distribution of the standard deviations of samples from a normal curve with the number of the individuals sampled reduced by a unit.

It is accordingly clear that in the problems considered by Messrs Welch and Kołodziejczyk the question of whether their σ_t (= our $\overline{\sigma}_{a,i}$) in a series of samples is the same can be treated without regard to the α and β of the regression line, while the α and β cannot be considered separately as they are correlated. We shall consider later how to deal with this distribution of $\sigma_{a,i}$. We now turn to the correlation surface

$$z''' = z_0''' \frac{1}{Q^{\frac{1}{2}(M+1)}} e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2 (1-\rho^2)}{s_2^2 Q}} [d\alpha d\beta] \quad \text{.....(xix).}$$

This can be integrated out with regard to α, and gives us

$$z^{\text{iv}} = \frac{z_0^{\text{iv}}}{Q^{\frac{1}{2}M}} = \frac{z_0^{\text{iv}}}{\left(1 - 2\rho\frac{\beta s_1}{s_2} + \beta^2 \frac{s_1^2}{s_2^2}\right)^{\frac{1}{2}M}}$$

$$= \frac{z_0^{\text{v}}}{\left\{\frac{s_2^2}{s_1^2}(1-\rho^2) + \left(\beta - \rho\frac{s_2}{s_1}\right)^2\right\}^{\frac{1}{2}M}} \quad \text{.....(xx).}$$

This is the distribution curve for the regression coefficients β of samples of size M drawn from a bivariate normal surface. It is a curve of my Type VII, symmetrical about the regression coefficient of the parent population, $\rho\sigma_2/\sigma_1$, and as M increases it approaches a normal distribution†. Remembering that $s_2/s_1 = \sigma_2/\sigma_1$ we remark that

$$\bar{\beta} = \rho\sigma_2/\sigma_1, \quad \sigma_\beta = \frac{1}{\sqrt{M-3}}\frac{\sigma_2}{\sigma_1}\sqrt{1-\rho^2} \quad \text{.....(xxi).}$$

The odd moments all vanish and $\mu_4/\mu_2^2 = 3\,(M-3)/(M-5)$.

For low values of M the moments of the β curve become infinite. For $M=3$ μ_2 is infinite, and for $M=4$ the value of μ_4 is infinite. Generally M must be $> n+2$ for the nth moment to be finite; i.e. if we desire the nth moment to be finite, the number in the sample on which the regression line is based must be $> n+2$.

Returning to (xix) we see that it may be written

$$z''' = z_0''' e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2}{\sigma_2^2 Q/M}} \frac{1}{Q^{\frac{1}{2}(M+1)}} \quad \text{.....(xxii).}$$

Since Q is a function only of β we see that for constant regression coefficients α will be distributed normally about the

$$\text{Mean } \alpha_\beta = m_2 - \beta m_1,$$

* *R.S. Proc.* Vol. 112 A, p. 4.

† I discovered this distribution in 1926 (see *R.S. Proc.* Vol. 112 A, pp. 5—7), but Mr Welch has pointed out to me the proof then given was not valid.

with
$$\sigma_{\alpha.\beta} = \frac{\sigma_2\sqrt{Q}}{\sqrt{M}} = \frac{\sigma_2}{\sqrt{M}}\left(1 - 2\rho\frac{\beta\sigma_1}{\sigma^2} + \beta^2\frac{\sigma_1^2}{\sigma_2^2}\right)^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{M}}(\sigma_2^2 - 2\rho\beta\sigma_1\sigma_2 + \beta^2\sigma_1^2)^{\frac{1}{2}} \quad \text{.....(xxiii).}$$

Thus we learn that the regression of α on β is linear, but the system is heteroscedastic, the variance being given by a hyperbola.

I have tried unsuccessfully to integrate (xxii) for β, so as to obtain the distribution curve for α. This I fear is unlikely to be achieved, but the moments of the α curve can be found, and if the reader be so inclined a curve of one of my types fitted from the first four moments. This is the method adopted in a similar problem by Mr Welch (p. 151), but in the case of samples based on few individuals, I feel doubtful how far this approximation to the true distribution curve of a variate may not be misleading. In the case of many individuals it would for practical purposes be adequate.

We will turn now to the moments of α. First we will find the value of z_0''' supposing the total frequency to be N. We have

$$N = z_0''' \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\frac{(\alpha - m_2 + m_1\rho(\sigma_2/\sigma_1) + m_1 R_2')^2}{\sigma_2^2 Q/M}} \frac{1}{Q^{\frac{1}{2}(M+1)}}\, d\alpha\, dR'_2,$$

where
$$Q = \frac{\sigma_1^2}{\sigma_2^2}\left(\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2) + R_2'^2\right)$$

and
$$R_2' = \beta - \rho\frac{\sigma_2}{\sigma_1}.$$

Integrating out with regard to α,

$$N = z_0''' \int_{-\infty}^{+\infty} \frac{\sqrt{2\pi}\sigma_2\sqrt{Q}}{\sqrt{M}} \frac{1}{Q^{\frac{1}{2}(M+1)}}\, dR_2'$$
$$= \frac{\sqrt{2\pi}\sigma_2 z_0'''}{\sqrt{M}} \int_{-\infty}^{+\infty} \frac{1}{Q^{\frac{1}{2}M}}\, dR_2' = \frac{\sqrt{2\pi}}{\sqrt{M}} \frac{\sigma_2^{M+1}}{\sigma_1^M z_0'''} \int_{-\infty}^{+\infty} \frac{dR_2'}{(c^2 + R_2'^2)^{\frac{1}{2}M}}$$

where
$$c = \frac{\sigma_2}{\sigma_1}\sqrt{1-\rho^2}.$$

Hence
$$N = z_0'''\sqrt{2}\pi\frac{\sigma_2^2}{\sigma_1}\frac{\sqrt{M}\,\Gamma(\frac{1}{2}(M-1))}{(1-\rho^2)^{\frac{1}{2}(M-1)}\,\Gamma(\frac{1}{2}M)},$$

or
$$z_0''' = \frac{N\sigma_1(1-\rho^2)^{\frac{1}{2}(M-1)}\sqrt{M}\,\Gamma(\frac{1}{2}M)}{\sqrt{2}\pi\sigma_2^2\,\Gamma(\frac{1}{2}(M-1))} \quad \text{.....(xxiv).}$$

While considering the value of the z_0's it is worth while writing down what value arises from the γ distribution—i.e.

$$z = z_0^{iv}\, e^{-\frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\gamma}{s_2}\right)^{M-3}[d\gamma],$$

where $s_2^2 = \sigma_2^2(1-\rho^2)/M$—of the three-way frequency surface.

Putting $v = \frac{1}{2}\frac{\gamma^2}{s_2^2}$, integrating from 0 to ∞, and taking the total frequency to be unity we find

$$z_0^{iv} = \frac{\sqrt{M}}{\sigma_2\sqrt{1-\rho^2}\,2^{\frac{1}{2}(M-4)}\,\Gamma(\frac{1}{2}M-1)},$$

or the distribution of γ, the array standard deviation, is given by

$$z = \frac{\sqrt{M}}{\sigma_2\sqrt{1-\rho^2}\,2^{\frac{1}{2}(M-4)}\,\Gamma(\frac{1}{2}M-1)}\,e^{-\frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\gamma}{s_2}\right)^{M-3} \quad \text{.....(xxv).}$$

We are now in a position to write down the complete three-way surface for the frequency distribution of α, β, γ. It is

$$z = \frac{N\sigma_1(1-\rho^2)^{\frac{1}{2}(M-2)}(M-2)\,M}{2^{\frac{1}{2}(M-1)}\,\pi\,\sigma_2{}^3\,\Gamma(\frac{1}{2}(M-1))}\,e^{-\frac{1}{2}\frac{(\alpha-m_2+\beta m_1)^2}{\sigma_2{}^2 Q/M}}\,\frac{1}{Q^{\frac{1}{2}(M+1)}}$$
$$\times\, e^{-\frac{1}{2}\frac{M\gamma^2}{\sigma_2{}^2(1-\rho^2)}}\left(\frac{\sqrt{M}\,\gamma}{\sigma_2\sqrt{1-\rho^2}}\right)^{M-3}[d\alpha\,d\beta\,d\gamma] \quad \text{.....(xxvi).}$$

For the two-way surface of α and β only

$$z = \frac{N\sigma_1(1-\rho^2)^{\frac{1}{2}(M-1)}\,\Gamma(\frac{1}{2}M)\sqrt{M}}{\sqrt{2}\,\pi\,\sigma_2{}^2\,\Gamma(\frac{1}{2}(M-1))}\,e^{-\frac{1}{2}\frac{(\alpha-m_2+\beta m_1)^2}{\sigma_2{}^2 Q/M}}\,\frac{1}{Q^{\frac{1}{2}(M+1)}}[d\alpha\,d\beta] \text{...(xxvii).}$$

The importance of these forms is that for any hypothesis as to α, β, γ we may require to find the best values to take for m_1, m_2, σ_1, σ_2, and ρ in order to give the most probable values to the observed quantities*.

We now return to the moments of α, applying formula (xxvii). It is necessary to find $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z\alpha\,d\alpha\,d\beta$ to obtain the mean of α, or $\bar{\alpha}$. Now

$$N\bar{\alpha} = z_0''' \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\{(\alpha-m_2+\beta m_1)+(m_2-\beta m_1)\}\,e^{-\frac{1}{2}\frac{(\alpha-m_2+\beta m_1)^2}{\sigma_2{}^2 Q/M}}\,\frac{1}{Q^{\frac{1}{2}(M+1)}}\,d\alpha\,d\beta,$$

and if we write $X = \alpha - m_2 + \beta m_1$,

$$N\bar{\alpha} = z_0''' \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\{X+(m_2-\beta m_1)\}\,e^{-\frac{1}{2}\frac{X^2}{\sigma_2{}^2 Q/M}}\,\frac{1}{Q^{\frac{1}{2}(M+1)}}\,dX\,d\beta,$$

the integration of the first term in the curled brackets with regard to X is zero, and of the second term is $\sqrt{2\pi}\sqrt{Q/M}\,\sigma_2 \times (m_2-\beta m_1)$. Accordingly

$$N\bar{\alpha} = z_0''' \sqrt{2\pi}\int_{-\infty}^{+\infty}\sqrt{\frac{Q}{M}}\,\sigma_2(m_2-\beta m_1)\,\frac{1}{Q^{\frac{1}{2}(M+1)}}\,d\beta$$
$$= \frac{z_0''' \sqrt{2\pi}}{\sqrt{M}}\,\sigma_2\int_{-\infty}^{+\infty}\left\{\left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right) - m_1 R_2'\right\}\frac{dR_2'}{\left(\frac{\sigma_1}{\sigma_2}\right)^M\left(\frac{\sigma_2{}^2}{\sigma_1{}^2}(1-\rho^2)+R_2'^2\right)^{\frac{1}{2}M}},$$

where $R_2' = \beta - \rho\frac{\sigma_2}{\sigma_1}$ as before.

* The method was first used by Gauss to demonstrate the arithmetic mean of observations and the best value of the standard deviation in the case of the normal curve. It was largely applied by Professor Filon and myself in dealing with the variability of frequency constants in samples (*Phil. Trans.* (1898), Vol. 198 A, pp. 231—33).

Since the R_2' curve is symmetrical, the integral involving the R_2' term in the curled brackets vanishes, and we have

$$N\bar{a} = \frac{z_0'''\sqrt{2\pi}}{\sqrt{M}}\sigma_2\left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right)\left(\frac{\sigma_2}{\sigma_1}\right)^M \int_{-\infty}^{+\infty} \frac{dR_2'}{\left(\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2) + R_2'^2\right)^{\frac{1}{2}M}}.$$

Take $R_2' = \frac{\sigma_2}{\sigma_1}\sqrt{1-\rho^2}\sqrt{\frac{v}{1-v}}$, and we find

$$\int_{-\infty}^{+\infty} \frac{dR_2'}{\left(\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2) + R_2'^2\right)^{M/2}} = \left(\frac{\sigma_1}{\sigma_2}\right)^{M-1} \frac{1}{(1-\rho^2)^{\frac{1}{2}(M-1)}} \int_0^1 v^{-\frac{1}{2}}(1-v)^{\frac{1}{2}(M-3)}\,dv$$

$$= \left(\frac{\sigma_1}{\sigma_2}\right)^{M-1} \frac{1}{(1-\rho^2)^{\frac{1}{2}(M-1)}} \frac{\Gamma(\frac{1}{2})\,\Gamma\frac{1}{2}(M-1)}{\Gamma(\frac{1}{2}M)},$$

and accordingly

$$N\bar{a} = z_0''' \frac{\sqrt{2}\pi}{\sqrt{M}} \frac{\sigma_2^2}{\sigma_1} \frac{1}{(1-\rho^2)^{\frac{1}{2}(M-1)}} \frac{\Gamma\frac{1}{2}(M-1)}{\Gamma(\frac{1}{2}M)}\left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right).$$

Substituting for z_0''' from (xxiv), we have

$$\bar{a} = m_2 - m_1\rho\frac{\sigma_2}{\sigma_1} \quad \text{.....(xxviii).}$$

This checks the value of z_0''', giving to the mean a the population value. But it is not obvious *à priori* that the mean value of the triple product $\bar{x}r\frac{\Sigma_2}{\Sigma_1}$ will take the population value.

We will now proceed to compute σ_a, the standard deviation of a,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z\alpha^2\,d\alpha = z_0''' \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{\alpha - m_2 + \beta m_1 + m_2 - \beta m_1\}^2 e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2}{\sigma_2^2 Q/M}} \times \frac{1}{Q^{\frac{1}{2}(M+1)}}\,d\alpha\,d\beta.$$

Writing $\alpha - m_2 + \beta m_1 = X$ as before, the curled bracket term gives us

$$X^2 + 2(m_2 - \beta m_1)X + (m_2 - \beta m_1)^2;$$

this has to be multiplied by $e^{-\frac{1}{2}\frac{X^2}{\sigma_2^2 Q/M}}$, and first integrated with regard to X. This gives us

$$\sqrt{2\pi}\sigma_2\sqrt{\frac{Q}{M}}\,\sigma_2^2 Q/M + 0 + (m_2 - \beta m_1)^2\sqrt{2\pi}\sigma_2\sqrt{\frac{Q}{M}},$$

and thus $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z\alpha^2\,d\alpha\,d\beta = \frac{\sqrt{2\pi}\sigma_2 z_0'''}{\sqrt{M}} \int_{-\infty}^{+\infty}\left\{\frac{\sigma_2^2 Q}{M} + (m_2 - \beta m_1)^2\right\}\frac{d\beta}{Q^{\frac{1}{2}M}},$$

and this after some rearranging and introduction of $R_2' = \beta - \rho\frac{\sigma_2}{\sigma_1}$

$$= \frac{\sqrt{2\pi}\sigma_2 z_0'''}{\sqrt{M}} \int_{-\infty}^{+\infty}\left\{\frac{\sigma_2^2(1-\rho^2)}{M} + \left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right)^2 - 2m_1 R_2'\left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right) + R_2'^2\left(\frac{\sigma_1^2}{M} + m_1^2\right)\right\}\left(\frac{\sigma_2}{\sigma_1}\right)^M \frac{dR_2'}{(c^2 + R_2'^2)^{\frac{1}{2}M}}.$$

The term in the curled brackets in R_2' will disappear on integration, since the R_2' frequency is symmetrical. Hence

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z\alpha^2\,d\alpha\,d\beta = \frac{\sqrt{2\pi}\,\sigma_2 z_0'''}{\sqrt{M}}\int_{-\infty}^{+\infty}\left\{\left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right) - m_1^2\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2)\right.$$
$$\left. + (c^2 + R_2'^2)\left(\frac{\sigma_1^2}{M} + m_1^2\right)\right\}\left(\frac{\sigma_2}{\sigma_1}\right)^M \frac{dR_2'}{(c^2+R_2'^2)^{\frac{1}{2}M}}$$

$$= \frac{\sqrt{2\pi}\,\sigma_2 z_0'''}{\sqrt{M}}\left(\frac{\sigma_2}{\sigma_1}\right)^M\left[\left\{\left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right) - m_1^2\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2)\right\}\int_{-\infty}^{+\infty}\frac{dR_2'}{(c^2+R_2'^2)^{\frac{1}{2}M}}\right.$$
$$\left. + \left(\frac{\sigma_1^2}{M} + m_1^2\right)\int_{-\infty}^{+\infty}\frac{dR_2'}{(c^2+R_2'^2)^{\frac{1}{2}(M+2)}}\right] \text{.....(xxix).}$$

But
$$\int_{-\infty}^{+\infty}\frac{dR_2'}{(c^2+R_2'^2)^{\frac{1}{2}M}} = \frac{\sqrt{\pi}}{c^{M-1}}\frac{\Gamma\left(\frac{1}{2}(M-1)\right)}{\Gamma\left(\frac{1}{2}M\right)},$$

and
$$\int_{-\infty}^{+\infty}\frac{dR_2'}{(c^2+R_2'^2)^{\frac{1}{2}(M-2)}} = \frac{c^2\sqrt{\pi}}{c^{M-1}}\frac{\Gamma\left(\frac{1}{2}(M-1)\right)}{\Gamma\left(\frac{1}{2}M\right)}\frac{M-2}{M-3}.$$

Substituting in (xxix) these values, and also from (xxiv) the value of z_0''', we have

$$\int_{-\infty}^{+\infty} z\alpha^2\,d\alpha\,d\beta = N(\sigma_a^2 + \bar{a}^2) = N\left[\left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right)^2 + \frac{\sigma_2^2(1-\rho^2)}{M-3}\left(1 - \frac{2}{M} + \frac{m_1^2}{\sigma_1^2}\right)\right].$$

Or, since $\bar{a} = m_2 - m_1\rho\dfrac{\sigma_2}{\sigma_1}$, we have

$$\sigma_a^2 = \frac{\sigma_2^2(1-\rho^2)}{M-3}\left(1 + \frac{m_1^2}{\sigma_1^2} - \frac{2}{M}\right) \text{.....(xxx).}$$

The approximate values given by me for *large* samples in 1913* were, for α and β,

$$\sigma_a^2 = \frac{\sigma_2^2(1-\rho^2)}{M}\left(1 + \frac{m_1^2}{\sigma_1^2}\right) \text{ and } \sigma_\beta^2 = \frac{\sigma_2^2}{\sigma_1^2}\frac{(1-\rho^2)}{M},$$

as against the exact values

$$\sigma_a^2 = \frac{\sigma_2^2(1-\rho^2)}{M-3}\left(1 + \frac{m_1^2}{\sigma_1^2} - \frac{2}{M}\right) \text{ and } \sigma_\beta^2 = \frac{\sigma_2^2}{\sigma_1^2}\frac{1-\rho^2}{M-3};$$

thus the former values are seen to be closely approximate for large samples.

To indicate the dependence of α and β, we will find their correlation coefficient, at the same time reminding the reader that while the regression curve of α on β is linear, that of β on α may be very far from being so. It is indeed so complicated as to be unmanageable.

Proceeding as before, we find

$$\frac{S(\alpha\beta)}{N} - \bar{\alpha}\bar{\beta} = -\frac{m_1}{\sigma_1}\frac{\sigma_2^2}{\sigma_1}\frac{1-\rho^2}{M-3} \text{ .....(xxxi),}$$

leading to
$$r_{\alpha\beta} = -\frac{\dfrac{m_1}{\sigma_1}}{\sqrt{1 + \dfrac{m_1^2}{\sigma_1^2} - \dfrac{2}{M}}} \text{ .....(xxxii).}$$

* *Biometrika*, Vol. IX. p. 9. See also *Phil. Trans.* (1898), Vol. 191 A, p. 245.

The corresponding values found for large samples in 1913* were

$$-\frac{m_1}{\sigma_1}\frac{\sigma_2^2}{\sigma_1}\frac{1-\rho^2}{M} \text{ and } -\frac{\frac{m_1}{\sigma_1}}{\sqrt{1+\frac{m_1^2}{\sigma_1^2}}}$$

respectively. These approximations are now justified by the determination of the accurate values for samples of any size.

One more theoretical point may be noted. If $\tilde{y}_x$ be the mean of the x array of y's, then

$$\tilde{y}_x = \alpha + \beta x,$$

and the equation $\tilde{y}_x - \bar{y} = \alpha - \bar{\alpha} + (\beta - \bar{\beta})x$ is exact.

Hence we have

$$\sigma^2_{\tilde{y}_x} = \sigma_\alpha^2 + x^2\sigma_\beta^2 + 2r_{\alpha\beta}\sigma_\alpha\sigma_\beta x$$

$$= \frac{1}{M-3}\left[\sigma_2^2(1-\rho^2)\left(1+\frac{m_1^2}{\sigma_1^2}-\frac{2}{M}\right)+\frac{x^2}{\sigma_1^2}\sigma_2^2(1-\rho^2)-2xm_1\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2)\right],$$

or

$$\sigma^2_{\tilde{y}_x} = \frac{\sigma_2^2(1-\rho^2)}{M-3}\left\{1+\frac{m_1^2}{\sigma_1^2}-\frac{2x}{\sigma_1}\frac{m_1}{\sigma_1}+\frac{x^2}{\sigma_1^2}-\frac{2}{M}\right\}.$$

Finally

$$\sigma^2_{\tilde{y}_x} = \frac{\sigma_2^2(1-\rho^2)}{M-3}\left(1+\frac{(x-m_1)^2}{\sigma_1^2}-\frac{2}{M}\right) \quad \text{.....(xxxiii).}$$

This gives the exact standard error of the mean of an array as found from a regression line. The value for large samples found in 1913 was

$$\sigma^2_{\tilde{y}_x} = \frac{\sigma_2^2(1-\rho^2)}{M}\left(1+\frac{(x-m_1)^2}{\sigma_1^2}\right),$$

which is accordingly justified.

We note one further point. Formula (xxxii) may be written

$$r_{\alpha\beta} = -\frac{1}{\sqrt{1+\left(1-\frac{2}{M}\right)\frac{\sigma_1^2}{m_1^2}}} \quad \text{.....(xxxii}^{bis}\text{),}$$

and accordingly we see that if the coefficient of variation of the x variate be very small the relation of α to β tends to become perfect (negatively); but if this coefficient of variation be large, the correlation tends to be small. Since coefficients of variation are in practice usually small, we see that the relation of α and β is close, and the regression curve of β on α will tend to be linear like that of α on β.

Starting with the variables α, β, γ, we have seen that γ is distributed according to a frequency curve which is absolutely independent of α and β, but that the latter are given by a bivariate frequency surface, exhibiting a high degree of

* *Biometrika*, Vol. IX. p. 10.

correlation. We propose to replace α, β, γ by three new variates which are mutually independent. These are as before

$$\left.\begin{aligned} X &= \frac{\alpha - m_2 + \beta m_1}{\sigma_2\sqrt{Q}/\sqrt{M}}, \text{ where } Q = \frac{\sigma_1^2}{\sigma_2^2}\{c^2 + (\beta - \bar{\beta})^2\}, \text{ and } c^2 = \frac{\sigma_2^2}{\sigma_1^2}(1 - \rho^2) \\ Y &= \frac{c^2}{c^2 + (\beta - \bar{\beta})^2}, \text{ and therefore } Y = (1 - \rho^2)/Q, \end{aligned}\right\}$$

and finally

$$Z = \frac{1}{2}\frac{\gamma^2}{s_2^2} = \frac{1}{2}\frac{M\gamma^2}{\sigma_2^2(1 - \rho^2)}$$

.....(xxxiv).

As to the limits they are independent. Those of Z depend only on γ and accordingly are from 0 to $+\infty$. Those of Y depend only on β, but it will have equal values when $\beta - \bar{\beta}$ takes plus or minus equal numerical values. Hence integrals for Y must be doubled, its limits are $Y = 1$, for $\beta = \bar{\beta}$, and $Y = 0$, for $\beta - \bar{\beta} = \pm\infty$.

Turning to X we may write it, having regard to (xxiii), in the form

$$X = \frac{\bar{y} - m_2 - \beta(\bar{x} - m_1)}{\sigma_{a \cdot \beta}} = \frac{\bar{y} - m_2 - \beta(\bar{x} - m_1)}{\sigma_{(\bar{y} - \beta\bar{x}) \cdot \beta}}.$$

Now $\bar{y} - m_2 - \beta(\bar{x} - m_1) = -(\tilde{y}_x - \bar{y}) + \beta(x - \bar{x}) + \tilde{y}_x - m_2 - \beta(x - m_1)$,

where $\tilde{y}_x$ is the mean of the sample array we are considering, and accordingly

$$\tilde{y}_x - \bar{y} - \beta(x - \bar{x}) = 0.$$

Again $\sigma_{(\bar{y} - \beta\bar{x}) \cdot \beta} = \sigma_{\bar{y} - \tilde{y}_x + \tilde{y}_x - \beta(\bar{x} - x + x)} = \sigma_{\tilde{y}_x - \beta x} = \sigma_{\tilde{y}_x - m_2 - \beta(x - m_1)}$,

since m_2 and m_1 do not vary for the sample from the particular population, which has given the regression coefficient β and the mean $\bar{x}$ and $\bar{y}$. Accordingly we have

$$X = \frac{\tilde{y}_x - m_2 - \beta(x - m_1)}{\sigma_{\tilde{y}_x - m_2 - \beta(x - m_1)}}.$$

Let us consider the geometrical meaning of X. $S(m_1, m_2)$ is the mean of any one of the k populations, and $T(\bar{x}, \bar{y})$ of the sample from this population with regression coefficient β (see figure, p. 239). AB is the regression line of this sample; CD is a parallel line through S of which the equation will be

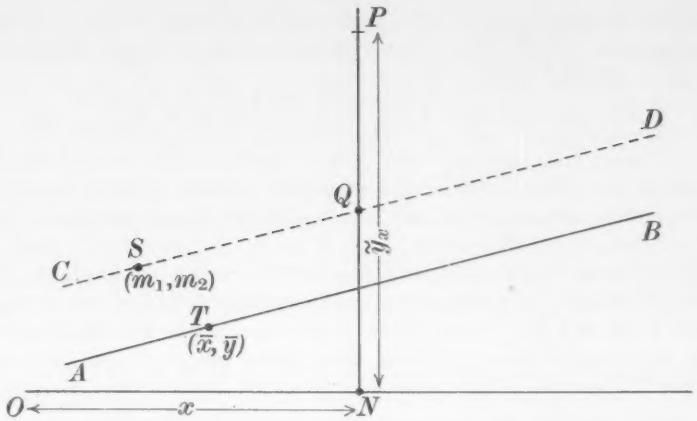
$$y - m_2 - \beta(x - m_1) = 0.$$

$ON = x$ gives the position of the array of y's for this sample of which the mean is $\tilde{y}_x$ at P. Now

$$PQ = PN - QN = \tilde{y}_x - \{m_2 + \beta(x - m_1)\} = \tilde{y}_x - m_2 - \beta(x - m_1).$$

Thus we have $$X = \frac{PQ}{\text{standard deviation of } PQ}$$ .....(xxxiv^{bis}).

But by hypothesis the arrays are to form normal distributions and accordingly fixing any particular sample by $\beta, \bar{x}, \bar{y}$ the limits of X are from $-\infty$ to $+\infty$.



We now proceed to transform our equation to the three-way surface from α, β, γ to X, Y, Z.

In order to transform the integral we must replace $d\alpha d\beta$ by

$$dX\,dY \begin{vmatrix} \frac{d\alpha}{dX}, & \frac{d\alpha}{dY} \\ \frac{d\beta}{dX}, & \frac{d\beta}{dY} \end{vmatrix} = dX\,dY \frac{d\alpha}{dX} \frac{d\beta}{dY},$$

since β is a function of Y only.

Now $\alpha = \sigma_2 \sqrt{Q/M}\, X + m_2 - \beta m_1$, where both Q and β can be expressed as a function of Y only. Accordingly

$$\frac{d\alpha}{dX} = \sigma_2 \sqrt{Q/M} \quad \text{.....(xxxv).}$$

Further since

$$Y = \frac{c^2}{c^2 + (\beta - \bar{\beta})^2},$$

$$\beta - \bar{\beta} = \pm c \sqrt{1/Y - 1},$$

$$\frac{d\beta}{dY} = \mp \tfrac{1}{2} c Y^{-\frac{3}{2}} (1 - Y)^{-\frac{1}{2}} \quad \text{.....(xxxvi),}$$

where we may drop the double sign on the understanding that we have to double the result if we integrate out for Y. We will now write down the transformation of (xxvi) remembering that $1/Q = Y/(1 - \rho^2)$, and that

$$d\alpha\, d\beta = \frac{1}{2} \frac{1}{\sqrt{M}} \sigma_2 c \sqrt{1 - \rho^2}\, Y^{-2} (1 - Y)^{-\frac{1}{2}}\, dX\, dY,$$

$$z = \frac{N \sigma_1 (1 - \rho^2)^{\frac{1}{2}(M-2)} (M - 2)\, M}{2^{\frac{1}{2}(M-1)}\, \pi \sigma_2^3\, \Gamma(\frac{1}{2}(M-1))} \frac{e^{-\frac{1}{2}X^2}}{\sqrt{M}}\, dX\, \tfrac{1}{2} \sigma_2 c \sqrt{1 - \rho^2} \left(\frac{Y}{1 - \rho^2}\right)^{\frac{1}{2}(M+1)}$$

$$\times Y^{-2} (1 - Y)^{-\frac{1}{2}}\, dY \frac{1}{\sqrt{2}} \frac{\sigma_2 \sqrt{1 - \rho^2}}{\sqrt{M}}\, 2^{\frac{1}{2}(M-3)}\, e^{-z}\, Z^{\frac{1}{2}(M-4)}\, dZ,$$

or

$$z\,dX\,dY\,dZ = \frac{N(M-2)}{2\sqrt{2\pi}\,\Gamma(\frac{1}{2}(M-1))}\,e^{-\frac{1}{2}X^2}\,\tfrac{1}{2}Y^{\frac{1}{2}(M-3)}(1-Y)^{-\frac{1}{2}}\,e^{-Z}\,Z^{\frac{1}{2}(M-4)}\,dX\,dY\,dZ$$

$$= N\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}X^2}\,dX \times \frac{\frac{1}{2}Y^{\frac{1}{2}(M-3)}(1-Y)^{-\frac{1}{2}}\,dY}{B(\frac{1}{2}(M-1),\frac{1}{2})} \times \frac{e^{-Z}\,Z^{\frac{1}{2}(M-4)}}{\Gamma(\frac{1}{2}(M-2))}\,dZ$$

.....(xxxvii).

Thus our three-way surface in α, β, γ has been reduced to the product of three frequency curves independent of each other; namely a normal curve in X, a Type I curve in Y and a Type III curve in Z.

Let us now consider how we may proceed. If we want to know whether on the k observed series we may suppose the array standard deviations are constant we calculate k Z's, and ask if these Z's are a random selection from the Z curve. To do this we calculate their probability integrals by means of the *Tables of the Incomplete* Γ-*Function*, and then find the value of P_{λ_n} by aid of the same tables*.

If we wish to know whether the k series have the same regression coefficient, we calculate the values of Y (or, if we prefer, β) for these k series, and by aid of the *Tables of the Incomplete* B-*Function* find the values of the k corresponding probability integrals, then by the *Tables of the Incomplete* Γ-*Function* again find P_{λ_n}. This may be done without first supposing the k-array standard deviations to be the same. If we suppose both the array standard deviations and the regression coefficients to be the same, then we have $2k$ probability integrals, and we can determine P_{λ_n} for the $2k$ as one set, instead of two sets of k.

Lastly if we want the equations to the regression lines to be the same we have to deal with X, which first introduces α_i, and ask whether the k values of X form a random sample from the X normal curve. We calculate the probability integrals of our k values of X from the normal curve probability integral table† and we thus have three sets of k probability integrals, or one set of $3k$, to test by the P_{λ_n} method. The advantage of this method is that in each of the tests of k X's, Y's and Z's we are working with the actual curve of distribution, and not with an empirical curve fitted by two moments.

Now I have said above that we must compute the observed values of the X's, Y's and Z's, but this is not possible unless we have a knowledge of the constants of the k parent populations. These must be obtained in the customary manner by choosing them so as to make the probability of the observed results a maximum. I have expressed the opinion that on the assumptions of common array standard deviations and normal array distributions we lose little, if anything, in generality by assuming the k populations to be all normal distributions. The determination of the constancy of the array standard deviations indicates that we must maximise the product of the k Type III values to find s_2 or $\sigma_2\sqrt{1-\rho^2}/\sqrt{M}$. This constant is not to vary for the k populations. If we now desire further that the regression coefficients should be constant for the k populations, we may do it in two ways: (a) By supposing each population has a different $\overline{\sigma}_{1,i}$, in which case the c_i^2 are all different. But if we have fixed the array standard deviation and the regression

* Or by aid of Miss F. N. David's Table, *Biometrika*, Vol. xxvi. pp. 7—11.

† *Tables for Statisticians and Biometricians*, Part I. pp. 2—8.

coefficient $\bar{\beta}_i$ of all surfaces as the same, this is only rendered possible by assuming that for our k populations, $\bar{\sigma}^2_{2,i} \propto 1/(1-\rho_i^2)$. This I have termed the relation (v), and find difficult of acceptance: see p. 228. Turning to the other alternative: (*b*) By supposing $\bar{\sigma}_{2,i}$ and $\bar{\rho}_i$ to have the same value for all the populations; then the fixing of $\bar{\beta}_i$ demands that $\bar{\sigma}_{1,i}$ shall be the same for all the populations, and we have c_i fixed. In this latter case, since σ_1, σ_2 and ρ are fixed for all populations it follows that our k populations may be treated as described by a single surface moved parallel to itself, but having a different mean for each population. If we accept (*a*) then $\sigma^2_{2,i}(1-\rho_i^2) = \text{constant} = \sigma^2_{2,i} - \bar{\beta}_i^2 \sigma^2_{1,i}$, and accordingly there is a hyperbolic relation between the two standard deviations—or a linear one between their variances—as singular and remarkable a relation among k populations as (v).

When we turn to the X distribution and consider the most likely values for $m_{1,i}$ and $m_{2,i}$ defining the x and y means of the ith population, we find, if there be only one array value sampled from that population, that the most likely values of $m_{1,i}$ and $m_{2,i}$ are not separately determinable, but that the best position for them is on the observed regression line of the ith population sample. This leads to $X_i = 0$, or the probability integral = ·5 for all k samples. Accordingly we reach $P_{\lambda_k} = \Gamma(\cdot 693{,}1472\sqrt{k}, k-1)$. For example: $P_{\lambda_4} = \cdot 3020$, $P_{\lambda_{25}} = \cdot 0489$, $P_{\lambda_{49}} = \cdot 0093$. Thus we require the number k of populations to be large before it indicates an improbable series of means when the mean of each population is placed in its most likely position, namely somewhere on the observed regression line of the sample. With a small number of populations, such as four, no discrimination against the hypothesis that they have independent means is likely to be provided by the X distribution. If we suppose that all the k populations may be represented by normal surfaces having the same σ_1, σ_2 and ρ, but differing means, we get no adequate test from the distribution of X, but if we suppose them to have the same mean, i.e. all the k samples to be drawn from the same normal surface, or possibly each sample from a different region of the same surface*, then we shall obtain a test from the X curve. We can again obtain a test from the X curve, if we break each i sample up into two or more subsamples, for in this case we have equations enough to determine $m_{1,i}$ and $m_{2,i}$. If we have s subsamples in the ith sample of the ith population, and we have already fixed c^2, $\bar{\beta}$ and the array variability, we shall have in order to determine $\bar{\alpha}_i$ and $m_{1,i}$ to minimise the expression

$$\sum_{t=0}^{s} \frac{\{\alpha_{i,t} - \bar{\alpha}_i + m_{1,i}(\beta_{i,t} - \bar{\beta})\}^2}{\sigma_1^2\{c^2 + (\beta_{i,t} - \bar{\beta})^2\}}$$

with regard to $\bar{\alpha}_i$ and $m_{1,i}$, which leads to the equations

$$\left.\begin{aligned} \bar{\alpha}_i \sum_{t=1}^{s} \frac{1}{c^2 + (\beta_{i,t} - \bar{\beta})^2} - m_{1,i} \sum_{t=1}^{s} \frac{(\beta_{i,t} - \bar{\beta})}{c^2 + (\beta_{i,t} - \bar{\beta})^2} &= \sum_{t=1}^{s} \frac{\alpha_{i,t}}{c^2 + (\beta_{i,t} - \bar{\beta})^2} \\ \bar{\alpha}_i \sum_{t=1}^{s} \frac{\beta_{i,t} - \bar{\beta}}{c^2 + (\beta_{i,t} - \bar{\beta})^2} - m_{1,i} \sum_{t=1}^{s} \frac{(\beta_{i,t} - \bar{\beta})^2}{c^2 + (\beta_{i,t} - \bar{\beta})^2} &= \sum_{t=1}^{s} \frac{\alpha_{i,t}(\beta_{i,t} - \bar{\beta})}{c^2 + (\beta_{i,t} - \bar{\beta})^2} \end{aligned}\right\} \quad \ldots(\text{xxxviii}).$$

* This is the condition (*b*) of p. 228.

Since $\bar{a}_i = m_{2,i} - \bar{\beta} \times m_{1,i}$, we clearly have equations to determine $m_{1,i}$ and $m_{2,i}$ the means of the ith parent population. If there be only one array for the sample the sums reduce to a single term, the factor $\dfrac{1}{c^2 + (\beta_{i,1} - \bar{\beta})^2}$ divides out of the first equation and the factor $\dfrac{\beta_{i,1} - \bar{\beta}}{c^2 + (\beta_{i,1} - \bar{\beta})^2}$ from the second, and both reduce to

$$\bar{a}_i - m_{1,i}(\beta_{i,1} - \bar{\beta}) = a_{i,1},$$

or

$$m_{2,i} - m_{1,i}\bar{\beta} - m_{1,i}(\beta_{i,1} - \bar{\beta}) = \bar{y}_i - \bar{x}_i\beta_{i,1},$$

which is $m_{2,i} - \bar{y}_i = \beta_{i,1}(m_{1,i} - \bar{x}_i)$, and is equivalent to saying that the most probable values of $m_{1,i}$ and $m_{2,i}$ lie on the regression line of the sample, but it does not further define their position.

There are innumerable hypotheses we can make with regard to the constants $m_{1,i}$, $m_{2,i}$, $\sigma_{1,i}$, $\sigma_{2,i}$, ρ_i of the k surfaces. The method of testing them is the same in all cases, namely to deduce the most likely values of these constants by the usual method of maximising the total probability of the observed event, subject to the conditions imposed on the $5k$ constants, and then apply the P_{λ_n} test of reasonableness to the k values of X, Y and Z.

I will now proceed to illustrate these methods on the data provided by Mr Wilsdon of the Building Research Station, which were used by Mr Welch in his paper, and which Mr Wilsdon has most kindly permitted me also to use. The raw data will be given here as we need to table more constants than are published by Mr Welch. We agree absolutely with Mr Welch's values of β_i and have adopted his values of the array variances; all the other constants except the latter have been worked out by Miss David and myself. The populations each represent eight tests of the crushing strengths of concrete and mortar in lbs. per sq. inch; the individual results represent different cements used in making the mortar and the concrete, the same cement being used for both. The four groups ($i = 1, 2, 3, 4$) represent tests made when the age of mortar and concrete was one day, three days, seven days and twenty-eight days respectively. The regression line is for concrete (y), on mortar (x).

Crushing Strength ($i=1$)		Crushing Strength ($i=2$)		Crushing Strength ($i=3$)		Crushing Strength ($i=4$)	
Normal Concrete	Dry Mortar	Normal Concrete	Dry Mortar	Normal Concrete	Dry Mortar	Normal Concrete	Dry Mortar
888	2632	2608	5847	4252	7497	5568	8947
1738	4932	4455	8165	5678	9358	7298	10658
256	1368	1560	3242	3189	4533	5568	6318
1914	4772	4199	7715	5362	8927	7550	11000
724	2334	2363	4477	3448	5452	5013	7165
2190	5680	4667	7505	5755	7970	6963	8972
834	2304	2200	4783	3335	6314	5860	8458
1696	4021	3476	6809	4492	8256	6138	9548

From this table the following constants were deduced:

Constant	Sample ($i=1$)	Sample ($i=2$)	Sample ($i=3$)	Sample ($i=4$)
$\bar{y}_i$	1280	3191	4438·875	6244·375
$\bar{x}_i$	3505·375	6067·875	7288·375	8883·25
$\sigma_{2,i}$	644·5122	1092·0895	994·9920	860·6380
$\sigma_{1,i}$	1448·3196	1656·1133	1593·9983	1491·5566
r_i	·986,649	·965,638	·892,629	·837,205
β_i	·43907	·63677	·55719	·48307
α_i	−259·0895	−672·8414	+377·873	+1953·1193
$\gamma_i^2=\sigma^2_{2,i}(1-r_i^2)$	3323	9541	7920	5926

As we have seen there are five independent constants for each population on the assumption that it may be treated as a normal population, namely, $m_{1,i}$, $m_{2,i}$, $\sigma_{1,i}$, $\sigma_{2,i}$, ρ_i. We may replace these by five equally independent constants, namely,

$$\bar{\beta}_i=\rho_i\sigma_{2,i}/\sigma_{1,i},\quad c_i^2=\sigma^2_{2,i}(1-\rho_i^2)/\sigma^2_{1,i},\quad \bar{\gamma}_i^2=\sigma^2_{2,i}(1-\rho_i^2),\quad m_{1,i},$$

and

$$\bar{\alpha}_i=m_{2,i}-\bar{\beta}_i m_{1,i}.$$

Hence the distribution curve of γ_i's in the ith population is from (xxv)

$$z_i=\frac{\sqrt{M}}{\bar{\gamma}_i 2^{\frac{1}{2}(M-1)}\,\Gamma(\frac{1}{2}M-1)}\,e^{-\frac{1}{2}\frac{M\gamma_i^2}{\bar{\gamma}_i^2}}\left(\frac{\gamma_i\sqrt{M}}{\bar{\gamma}_i}\right)^{M-3}[d\gamma_i],$$

where the constants $\bar{\gamma}_i$ will *not* occur in any other of the three independent distributions. Thus in determining the most likely value of $\bar{\gamma}_i$ we need only consider this the Z distribution. Now the hypothesis we are to test is whether the array standard deviation is the same for all k populations, or $\bar{\gamma}_i=\bar{\gamma}$ a quantity independent of i. Accordingly we must make

$$P=\frac{(\sqrt{M})^k}{\bar{\gamma}^k}\,\frac{1}{2^{\frac{1}{2}k(M-4)}\,(\Gamma(\frac{1}{2}M-1))^k}\,e^{-\frac{M}{2}\overset{k}{S}_1\left(\frac{\gamma_i}{\bar{\gamma}}\right)^2}\underset{i=1\text{ to }k}{\text{Product}}\left(\frac{\sqrt{M}\gamma_i}{\bar{\gamma}}\right)^{M-3}d\gamma_1\ldots d\gamma_k$$

a maximum with regard to $\bar{\gamma}$.

Taking logarithmic differentials we have

$$-\frac{k}{\bar{\gamma}}+M\overset{k}{S}_1\frac{(\gamma_i^2)}{\bar{\gamma}^3}-\frac{k\,(M-3)}{\bar{\gamma}}=0,$$

or

$$\bar{\gamma}^2=\frac{M}{M-2}\overset{k}{S}_1\frac{(\gamma_i^2)}{k}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxxix)}.$$

Hence the most likely value of the array standard deviation is given by

$$\sigma^2_{2,i}(1-\rho_1^2)=\frac{M}{M-2}\quad(\text{mean value of }\gamma_i^2),$$

and the quantity we have called $s_{2,i}$ by

$$s^2_{2,i}=\frac{1}{M-2}\quad(\text{mean value of }\gamma_i^2)\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(xl)}.$$

Here we are met by the old difficulty to which we have referred, namely, we have to suppose that

$$\sigma^2_{z,i} \propto \frac{1}{1-\rho_i^2} \quad \ldots\ldots \quad \ldots\ldots\ldots\

are as follows:

$$u_1 = \cdot 861{,}9401, \quad u_2 = 2\cdot 474{,}8031, \quad u_3 = 2\cdot 054{,}3382, \text{ and } u_4 = 1\cdot 537{,}1222.$$

From these by merely linear interpolation into the Γ-function table which will suffice practically we have, if $p' = 1 - p$,

$$p_1 = \cdot 189{,}6896, \quad \log p_1 = \bar{1}\cdot 278{,}0435, \quad p_1' = \cdot 810{,}3104, \quad \log p_1' = \bar{1}\cdot 908{,}6514,$$
$$p_2 = \cdot 800{,}7515, \quad \log p_2 = \bar{1}\cdot 903{,}4977, \quad p_2' = \cdot 199{,}2485, \quad \log p_2' = \bar{1}\cdot 299{,}3952,$$
$$p_3 = \cdot 689{,}4926, \quad \log p_3 = \bar{1}\cdot 838{,}5297, \quad p_3' = \cdot 310{,}5074, \quad \log p_3' = \bar{1}\cdot 492{,}0720,$$
$$p_4 = \cdot 496{,}8960, \quad \log p_4 = \bar{1}\cdot 696{,}2655, \quad p_4' = \cdot 503{,}1040, \quad \log p_4' = \bar{1}\cdot 701{,}6578.$$

Hence

$$\lambda_n = \log p_1 p_2 p_3 p_4 = -1\cdot 283{,}6636, \quad \lambda_n' = \log p_1' p_2' p_3' p_4' = -1\cdot 598{,}2236,$$

and $\dfrac{-\lambda_n}{\sqrt{n}\log_{10} e}$, since $n = 4$, gives us

$$-\frac{\lambda_n}{\cdot 868{,}5890} = 1\cdot 477{,}8725 \text{ and } -\frac{\lambda_n'}{\cdot 868{,}5890} = 1\cdot 840{,}0228,$$

leading to $P_{\lambda_n} = I\,(1\cdot 477{,}8725,\ 3)$, and $P_{\lambda_n'} = I\,(1\cdot 840{,}0228,\ 3)$,

or $P_{\lambda_n} = \cdot 34288$, $P_{\lambda_n'} = \cdot 50149$,

$Q_{\lambda_n} = 1 - P_{\lambda_n} = \cdot 65712$, $Q_{\lambda_n'} = P_{\lambda_n'} = \cdot 49851$.

Thus whether we reckon our probability integrals from the left or right end of our frequency curve the same conclusion must be drawn, that the four array standard deviations form a most reasonable sample from a set of normal populations having a common array standard deviation

$$\sigma_2\sqrt{1-\rho^2} = \sqrt{\frac{M}{M-2}}\,(\text{mean value of } \gamma_i^2)^{\frac{1}{2}} = \sqrt{8903\cdot 3333}$$
$$= 94\cdot 3575, \text{ with standard error } 23\cdot 1365,$$

as against the observed values, 57·6455, 97·6780, 88·9944 and 76·9805. The only deviation claiming special consideration is the first, which is about 1·59 times the standard error, and a deviation as small as or smaller than this would occur in about 19 in 100 trials*, or about ·76 times in four trials; it occurs once. Mr Welch gives a far higher improbability, = ·0126, for the four values being a random sample from his curve of distribution of the γ's. I think the difference arises from our methods. I have given, on the assumption that the surfaces are normal, what I think is the real curve of distribution of the γ's from which a little sample of four has been taken. Mr Welch obtains a compound criterion and selects a curve of my Type I from the first two moments to describe the variation of his criterion. While I believe for considerable material this method might give adequate results, I have grave doubts whether, if there really exist only four observed values, it may not possibly be misleading. At any rate, if my arithmetic be correct, I cannot raise objection to these observed values being a random sample from four *normal* frequency surfaces giving exact curves of distribution, and I will start on the assumption that they do.

* See p_1 above.

We now pass to the question whether if $\sigma_{2,i}\sqrt{1-\rho_i^2}$ be treated as a constant for the arrays of all the k populations, the values of $R_2=\rho_i\sigma_{2,i}/\sigma_{1,i}$ can be considered as a random sample from a series Π_1, Π_2, ... Π_k of normal populations having the same ρ_i, $\sigma_{2,i}$, but not necessarily the same $\sigma_{1,i}$, $\bar{x}_i$ and $\bar{y}_i$, i.e. the α_i's and $\bar{x}_i$'s are not necessarily the same. In order to get rid of α_i we must start from the distribution of R_2 or β_i in each of the individual populations. This will be of the form (xxi), or

$$z=\frac{z_0}{\left\{\frac{\sigma^2_{2,i}}{\sigma^2_{1,i}}(1-\rho_i^2)+\left(\beta_i-\rho_i\frac{\sigma_{2,i}}{\sigma_{1,i}}\right)^2\right\}^{\frac{1}{2}M}} \quad \text{.....(xlii).}$$

Here the quantity $\sigma^2_{2,i}(1-\rho_i^2)$ has already been supposed constant $=\overline{\gamma}$ for all populations. Our hypothesis to be tested is now that $\rho_i\frac{\sigma_{2,i}}{\sigma_{1,i}}$ is constant $=\bar{\beta}$ for all populations. Putting $c_i^2=\frac{\sigma^2_{2,i}(1-\rho_2^2)}{\sigma^2_{1,i}}=\frac{\overline{\gamma}^2}{\sigma^2_{1,i}}$, we can suppose either (a) that c_i^2 varies from population to population, or (b) that $c_i^2=c^2=$ a constant for all populations. Let us see whether one or other of these hypotheses is the more probable. We may write (xlii) in the form

$$z_i=\frac{Nc_i^{M-1}\Gamma\left(\frac{1}{2}M\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(M-1)\right)}\frac{1}{\{c_i^2+(\beta_i-\bar{\beta})^2\}^{\frac{1}{2}M}} \quad \text{.....(xliii}^{bis}\text{).}$$

(a) Since c_i only occurs in z_i the equation to determine its most likely value is

$$\frac{M-1}{2c_i^2}=\tfrac{1}{2}M\frac{1}{c_i^2+(\beta_i-\bar{\beta})^2}, \quad \text{or} \quad c_i^2=(M-1)(\beta_i-\bar{\beta})^2 \quad \text{.....(xliii),}$$

and for $\bar{\beta}$ occurring in all the k populations

$$\sum_{i=0}^{i=k}\frac{\beta_i-\bar{\beta}}{c_i^2+(\beta_i-\bar{\beta})^2}=0 \quad \text{.....(xliv),}$$

which by (xliii) gives, for the determination of $\bar{\beta}$,

$$\sum_{i=0}^{i=k}\frac{1}{\beta_i-\bar{\beta}}=0 \quad \text{.....(xlv).}$$

Assume $\bar{\beta}_0$ an approximate value of $\bar{\beta}$ and let $\bar{\beta}=\bar{\beta}_0+\epsilon$, where ϵ is supposed small, then we find

$$\epsilon=-\sum_{i=0}^{i=k}\frac{1}{\beta_i-\bar{\beta}_0}\Big/\sum_{i=0}^{i=k}\frac{1}{(\beta_i-\bar{\beta}_0)^2} \quad \text{.....(xlvi).}$$

A good taking-off point is to assume $\bar{\beta}_0$ to be the mean value of the k β_i's. This in our example is ·529,025, and substituting the values of β_i in (xlvi) we obtain $\epsilon=-$ ·006,126. Thus $\bar{\beta}=\bar{\beta}_0+\epsilon=$ ·522,899, not a very great change on $\bar{\beta}_0$, and for practical purposes needing no further approximation. We have now to consider the values taken by

$$Y_i=\frac{c_i^2}{c_i^2+(\beta_i-\bar{\beta})^2},$$

but this equals in every case $(M-1)/M$. Accordingly we have, since for $i=1$ and 4, $\beta_i - \bar{\beta}$ is negative, and for $i=2$ and 4, $\beta_i - \bar{\beta}$ is positive,

$$p_1 = p_4 = \tfrac{1}{2} I_{(M-1)/M}(3{\cdot}5,\ \tfrac{1}{2}),$$

and

$$p_2 = p_3 = 1 - \tfrac{1}{2} I_{(M-1)/M}(3{\cdot}5,\ \tfrac{1}{2}),$$

as the probability integrals derived from the Type III curve

$$z = z_0 Y_i^{\frac{1}{2}(M-3)}(1-Y_i)^{-\frac{1}{2}},$$

$I_x(p, q)$ being the incomplete Beta function. Looking up these values in the Table of that function, we find for $(M-1)/M = {\cdot}875$,

$$p_1 = p_4 = {\cdot}175{,}3968, \text{ and } p_2 = p_3 = {\cdot}824{,}6032.$$

It follows that $\lambda_n = \log(p_1 p_2 p_3 p_4) = -1{\cdot}679{,}4666$

and $$-\frac{\lambda_n}{\sqrt{n}\log_{10} e}, \text{ since } n=4, \ = 1{\cdot}932{,}176.$$

Accordingly $P_{\lambda_n} = I(1{\cdot}932{,}176, 3) = {\cdot}53949$

and $Q_{\lambda_n} = {\cdot}46051.$

There is no improbability therefore in supposing each of our four sets drawn from populations having the same regression coefficient, but different values of the quantity $c_i^2 = \sigma_2^2(1-\rho^2)/\sigma_{1,i}^2$.

It may be asked why we have not included the c_i^2 and $\bar{\beta}$ terms in X, when maximising the probability of the total observed data. The answer is that the maximising of X with regard to $\bar{\alpha}_i$ and $m_{1,i}$ causes the vanishing of the factor $\{\alpha_i - \bar{\alpha}_i + m_{1,i}(\beta_i - \bar{\beta})\}^2$ in X and so the other factor containing c_i^2 and $\bar{\beta}$ disappears with it (see our p. 241).

(*b*) Now on the other hypothesis we have for the four c_i^2's

$$c_1^2 = {\cdot}056{,}643, \quad c_2^2 = {\cdot}081{,}263, \quad c_3^2 = {\cdot}005{,}553, \text{ and } c_4^2 = {\cdot}014{,}783,$$

and these c_i^2's do not at first sight look like the result of random sampling from a common value. But if they were, this would mean that $\sigma_{1,i}$ would be the same for all four populations. The $\sigma_{1,i}$ values are given in the Table on p. 243, and *à priori* they do not look so improbable as a selection of four values of a standard deviation from four bivariate normal surfaces with the same σ_1, that is to say four random values taken from the frequency curve

$$z = \frac{N\sqrt{M}}{\sigma_1 2^{\frac{1}{2}(M-3)}\Gamma(\frac{1}{2}(M-1))} e^{-\frac{1}{2}M\left(\frac{\sigma_{1,i}}{\sigma_1}\right)^2} \left(\frac{\sqrt{M}\sigma_{1,i}}{\sigma_1}\right)^{M-2} [d\sigma_{1,i}] \ldots\ldots \text{(xlvii)}.$$

This leads us to the most likely value $\bar{\sigma}_1^2$ for σ_1^2, namely,

$$\bar{\sigma}_1^2 = \frac{M}{M-1} \frac{S}{i=1}^{i=k} \frac{(\sigma_{1,i}^2)}{k} = \frac{M}{M-1} \text{ (mean value of observed } x \text{ variances)}$$

$$= 1656{\cdot}6673 \text{ for our data, giving } \bar{\sigma}_1 = 407{\cdot}0218.$$

Taking $$v_i = \tfrac{1}{2}M\left(\frac{\sigma_{1,i}}{\sigma_1}\right)^2,$$

we find $$p_i = I\left(\frac{v_i}{\sqrt{\frac{1}{2}(M-1)}}, \tfrac{1}{2}(M-3)\right) = I\left(\frac{v_i}{\sqrt{3{\cdot}5}}, 2{\cdot}5\right).$$

Hence we obtain $p_1 = {\cdot}473{,}364$, $p_1' = {\cdot}526{,}636$,

$p_2 = {\cdot}666{,}665$, $p_2' = {\cdot}333{,}335$,

$p_3 = {\cdot}611{,}974$, $p_3' = {\cdot}388{,}026$,

$p_4 = {\cdot}515{,}390$, $p_4' = {\cdot}484{,}610$,

which give us $\lambda_n = -1{\cdot}002{,}028$, $\lambda_n' = -1{\cdot}481{,}355$,

and $P_{\lambda_n} = {\cdot}2013$, $P_{\lambda_n'} = {\cdot}4439$,

$Q_{\lambda_n} = {\cdot}7987$, $Q_{\lambda_n'} = {\cdot}5561$.

Thus, according to the P_{λ_n} criterion, there is no improbability in the four values of $\sigma_{1,i}$ being random samples from parent populations having a common $\sigma_1 = 1656{\cdot}6673$.

Returning, however, to our different c_i^2's and accordingly our different $\sigma_{1,i}^2$'s we have to find for those c_i^2's and the corresponding $\bar{\beta}$ ($= {\cdot}522{,}899$), whether the resulting four Y_i's given by

$$Y_i = \frac{c_i^2}{c_i^2 + (\beta_i - \bar{\beta})^2}$$

provide an improbable set of probability integrals. We have just seen (p. 247) that they give

$$P_{\lambda_n} = P_{\lambda_n'} = {\cdot}5395, \quad Q_{\lambda_n} = Q_{\lambda_n'} = {\cdot}4605.$$

Comparing the latter with $Q_{\lambda_n'} = {\cdot}5561$ as the more stringent series above, we see that to suppose the four surfaces to have different $\sigma_{1,i}$'s is a somewhat more probable hypothesis than supposing them to have a common σ_1, but the probability of the latter is so considerable that it certainly would not prevent us from accepting the hypothesis that all four populations have the same σ_1, σ_2 and ρ.

It may be objected, however, that for the proper determination of σ_1 we ought to have taken into consideration the general effect of fixing c and $\bar{\beta}$, and that we ought in determining σ_1 to proceed by maximising first the combined probability with regard to c and $\bar{\beta}$. This we will now proceed to do. The curve for the Y variate for a single population is, if as before, $c^2 = \sigma_2^2(1-\rho^2)/\sigma_1^2$ and $\bar{\beta} = \rho\sigma_2/\sigma_1$,

$$z = \frac{Nc^{M-1}\Gamma(\frac{1}{2}M)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}(M-1))} \frac{1}{\{c^2 + (\beta_i - \bar{\beta})^2\}^{\frac{1}{2}M}} \quad \text{.....(xlviii),}$$

and accordingly the probability to be maximised is

$$P = \frac{c^{k(M-1)}}{\pi^{\frac{1}{2}k}} \frac{(\Gamma(\frac{1}{2}M))^k}{\{\Gamma(\frac{1}{2}(M-1))\}^k} \prod_1^k \frac{1}{\{c^2 + (\beta_i - \bar{\beta})^2\}^{\frac{1}{2}M}};$$

hence differentiating with regard to c and $\bar{\beta}$ we find

$$\frac{k(M-1)}{c} - Mc \sum_1^k \frac{1}{\{c^2+(\beta_i-\bar{\beta})^2\}},$$

or

$$\frac{M-1}{M}\frac{1}{c^2} = \frac{1}{k}\sum_1^k \frac{1}{\{c^2+(\beta_i-\bar{\beta})^2\}} \quad \text{.....(xlix),}$$

and

$$\sum_1^k \left(\frac{\beta_i-\bar{\beta}}{c^2+(\beta_i-\bar{\beta})^2}\right) = 0 \quad \text{.....(l).}$$

These are the equations from which c^2 and $\bar{\beta}$ are to be found. Clearly if M be large and β_i not very different from $\bar{\beta}$, that is σ_β small (see p. 246), we can start with $\bar{\beta}$ = mean β_i as a first approximation.

Again, if M be large then (xlviii) approaches closely a normal curve, and the constants c^2 and $\bar{\beta}$ can be found directly without approximation*. When M is very small, as in the present case, we can only proceed by successive approximation. Let us suppose approximations c_0^2 and $\bar{\beta}_0$ have been found to c^2 and $\bar{\beta}$, and that $c_0^2+\zeta = c^2$ and $\bar{\beta}_0+\xi = \bar{\beta}$, where ξ and ζ are small quantities. Neglecting their squares, our equations for finding ξ and ζ are from (xlii^{bis})

$$\sum_1^k \frac{(\beta_i-\bar{\beta}_0)}{c_0^2+(\beta_i-\bar{\beta}_0)^2} = \xi\sum_1^k \frac{c_0^2-(\beta_i-\bar{\beta}_0)^2}{\{c_0^2+(\beta_i-\bar{\beta}_0)^2\}^2} + \zeta\sum_1^k \frac{(\beta_i-\bar{\beta}_0)}{\{c_0^2+(\beta_i-\bar{\beta}_0)^2\}^2} \quad \text{.....(li),}$$

$$k\frac{M-1}{M} - \sum_1^k \frac{c_0^2}{c_0^2+(\beta_i-\bar{\beta}_0)^2} = \xi\sum_1^k \frac{2c_0^2(\beta_i-\bar{\beta}_0)}{\{c_0^2+(\beta_i-\bar{\beta}_0)^2\}^2} + \zeta\sum_1^k \frac{(\beta_i-\bar{\beta}_0)^2}{\{c_0^2+(\beta_i-\bar{\beta}_0)^2\}^2} \quad \text{...(lii).}$$

In our case, $k=4$, $M=8$, and the mean value of the four observed β_i's, i.e. $\beta_1 = \cdot43907$, $\beta_2 = \cdot63677$, $\beta_3 = \cdot55719$ and $\beta_4 = \cdot48307$, is $\cdot529025$. We will start then with $\bar{\beta}_0 = \cdot53$, which nearly corresponds to this value, and $c_0^2 = \cdot04$, and compute our coefficients in (li) and (lii). We have

$$\sum_1^k \frac{(\beta_i-\bar{\beta}_0)}{c_0^2+(\beta_i-\bar{\beta}_0)^2} = -\cdot2512{,}0918, \quad \sum_1^k \frac{c_0^2}{\{c_0^2+(\beta_i-\bar{\beta}_0)^2\}^2} = 78\cdot8686{,}5354,$$

$$\sum_1^k \frac{(\beta_i-\bar{\beta}_0)^2}{\{c_0^2+(\beta_i-\bar{\beta}_0)^2\}^2} = 9\cdot5458{,}5438, \quad \sum_1^k \frac{(\beta_i-\bar{\beta}_0)}{\{c_0^2+(\beta_i-\bar{\beta}_0)^2\}^2} = -8\cdot5823{,}6398,$$

$$\sum_1^k \frac{c_0^2}{c_0^2+(\beta_i-\bar{\beta}_0)^2} = 3\cdot5365{,}8033, \quad k\frac{M-1}{M} = 3\cdot5.$$

Thus there result the numerical equations for ξ, ζ,

$$-\cdot2512{,}0918 = 69\cdot3227{,}9916\,\xi - 8\cdot5823{,}6398\,\zeta,$$

$$-\cdot0365{,}8033 = -\cdot6865{,}8912\,\xi + 9\cdot5458{,}5438\,\zeta,$$

leading to

$$\xi = -\cdot0041{,}3500, \quad \zeta = -\cdot0041{,}2948,$$

and thus to a second approximation

$$\bar{\beta} = \bar{\beta}_0 + \xi = \cdot53 - \cdot004{,}135 = \cdot525{,}865,$$

$$c^2 = c_0^2 + \zeta = \cdot04 - \cdot004{,}129 = \cdot035{,}871.$$

* In the *Annals of Eugenics*, Vol. v. p. 408 *et seq.* I have dealt with a case of M large, but varying in magnitude from one population to a second, replacing (xlviii) by a normal curve.

It would be easy to improve these slightly by a further approximation, but they are adequate for our purpose. They show that the normal populations from which we suppose the samples Π_1, Π_2, ... Π_4 selected have a regression coefficient in common most probably of the value ·525,865, which is strikingly near the value found for $\bar{\beta}$ on the hypothesis that the populations have independent $\sigma^2_{1,i}$'s—i.e. ·522,899. Further these populations have, as we have seen, the same array variance $\sigma_2^2(1-\rho^2) = 8903{\cdot}3333$ (see p. 245), but the value of

$$c^2 = \frac{\sigma_2^2(1-\rho^2)}{\sigma_1^2} = \frac{8903{\cdot}3333}{\sigma_1^2} = {\cdot}035{,}871.$$

Accordingly $$\sigma_1^2 = \frac{8903{\cdot}3333}{{\cdot}035{,}871} = 248{,}204{\cdot}21,$$

or $$\sigma_1 = 498{\cdot}201^*.$$

Now this result is of considerable interest. It is of course not the standard deviation of the values of x considered in the experiments; it is the most likely value of the standard deviation of x in the several normal surfaces with fixed c and $\bar{\beta}$ from which the observations are supposed drawn. It seems to me that σ_1 has a most probable value, the moment we drop the unacceptable (v). But we can now proceed further to find the probable values of σ_2 and ρ. We have

$$c^2 = \frac{\sigma_2^2}{\sigma_1^2} - \rho^2\frac{\sigma_2^2}{\sigma_1^2} = \frac{\sigma_2^2}{\sigma_1^2} - \bar{\beta}^2,$$

or $$\sigma_2^2 = \sigma_1^2(c^2 + \bar{\beta}^2) = 248{,}204{\cdot}21\,({\cdot}035{,}871 + {\cdot}276{,}534) = 77540{\cdot}2362,$$

and $$\sigma_2 = 278{\cdot}460{,}475.$$

Accordingly

$$\rho = \bar{\beta} \times \sigma_1/\sigma_2 = {\cdot}525{,}865 \times 498{\cdot}201/278{\cdot}460{,}475 = {\cdot}94084$$

gives the likely correlation.

We have now ascertained the σ_1, σ_2 and ρ of the most probable surfaces from which the regression lines have been ascertained. The next step is to consider whether β_1, β_2, β_3 and β_4 are probable values to be drawn from the now known curve of distribution of β. This curve is given in (xlviii), where we can insert the probable values of c^2 and β. The values of $\beta_i - \bar{\beta}$ are $\beta_1 - \bar{\beta} = -{\cdot}086{,}795$, $\beta_2 - \bar{\beta} = +{\cdot}110{,}905$, $\beta_3 - \bar{\beta} = +{\cdot}031{,}335$, $\beta_4 - \bar{\beta} = -{\cdot}042{,}795$.

From these and $c^2 = {\cdot}035{,}871$, we have to calculate

$$v_i = \frac{(\beta_i - \bar{\beta})^2}{c^2 + (\beta_i - \bar{\beta})^2} \quad \text{.....(liii),}$$

giving $v_1 = {\cdot}173{,}563$, $v_2 = {\cdot}255{,}339$, $v_3 = {\cdot}026{,}627$, $v_4 = {\cdot}048{,}575$.

By the above transformation of β_i to v_i, the proportional area of the distribution curve of β up to the value β_i from left to right, or the probability integral

$$p_i = \tfrac{1}{2}\{1 \pm I_{v_i}(\tfrac{1}{2}, 3{\cdot}5)\},$$

* The value of σ_1 without any constraint on the four populations to have the same $\bar{\beta}$ and c was 407·0218, which marks the considerable influence of the restraints (see p. 247).

where the plus sign corresponds to $\beta_i > \bar{\beta}$ and the minus sign to $\beta_i < \bar{\beta}$. These values of the incomplete Beta function must be found from the *Tables of the Incomplete Beta Function*, using as sufficient for the present purpose linear interpolation. But the Tables give $I_x(3{\cdot}5, \frac{1}{2})$, so that we must write

$$I_{v_i}(\tfrac{1}{2}, 3{\cdot}5) = {\cdot}1 - I_{1-v_i}(3{\cdot}5, \tfrac{1}{2}),$$

or

$$p_i = 1 - \tfrac{1}{2} I_{1-v_i}(3{\cdot}5, \tfrac{1}{2}), \quad \text{if } \beta_i > \bar{\beta},$$
$$= \tfrac{1}{2} I_{1-v_i}(3{\cdot}5, \tfrac{1}{2}), \quad \text{if } \beta_i < \bar{\beta}.$$

We deduce accordingly the following values:

$$p_1 = \quad \tfrac{1}{2} I_{{\cdot}838,502}(3{\cdot}5, 0{\cdot}5) = {\cdot}13238, \quad p_1' = {\cdot}86762;$$
$$p_2 = 1 - \tfrac{1}{2} I_{{\cdot}744,661}(3{\cdot}5, 0{\cdot}5) = {\cdot}91734, \quad p_2' = {\cdot}08266;$$
$$p_3 = 1 - \tfrac{1}{2} I_{{\cdot}973,379}(3{\cdot}5, 0{\cdot}5) = {\cdot}66182, \quad p_3' = {\cdot}33818;$$
$$p_4 = \quad \tfrac{1}{2} I_{{\cdot}951,425}(3{\cdot}5, 0{\cdot}5) = {\cdot}28457, \quad p_4' = {\cdot}71543;$$

$$\lambda_n = -\log(p_1 p_2 p_3 p_4) = -1{\cdot}640{,}7183, \quad \lambda_n' = -\log(p_1' p_2' p_3' p_4') = -1{\cdot}760{,}6600.$$

Dividing these values by $\sqrt{n} \log_{10} e = {\cdot}868{,}5890$, we have

$$P_{\lambda_n} = I(1{\cdot}888{,}947, 3), \qquad P_{\lambda_n'} = I(2{\cdot}027{,}035, 3),$$

whence from the *Tables of the Incomplete* Γ-*Function* we deduce by linear interpolation as adequate for our present purposes

$$P_{\lambda_n} = {\cdot}5219, \qquad P_{\lambda_n'} = {\cdot}5768,$$
$$Q_{\lambda_n} = 1 - P_{\lambda_n} = {\cdot}4781, \qquad Q_{\lambda_n'} = 1 - P_{\lambda_n'} = {\cdot}4232.$$

It is clear from these results that the four values of the regression coefficient provided might easily have been randomly selected from the distribution curve of samples of 8 drawn from normal bivariate surfaces differing only in their means. But we can hardly draw the inference that the distributions of the populations Π_1, Π_2, Π_3, Π_4 *are* accordingly normal, because the array standard deviations and the regression coefficients are capable of being considered random samples from the corresponding appropriate Type III and Type I curves. They, consisting of only *four* values each, may be shown to be equally random samples from normal curves or rectangles. But the method which, I think, would be legitimate when applied to larger samples can be illustrated on this example.

We now turn back to formula (xxvii) for the bivariate surface of α and β, namely,

$$z = \frac{N\sigma_1 (1-\rho^2)^{\frac{1}{2}(M-1)} \Gamma(\frac{1}{2}M) \sqrt{M}}{\sqrt{2}\,\pi\,\sigma_2^2\,\Gamma(\frac{1}{2}(M-1))} e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2}{\sigma_2^2 Q/M}} \frac{1}{Q^{\frac{1}{2}(M+1)}},$$

and

$$Q = 1 - \frac{2\rho\beta\sigma_1}{\sigma_2} + \beta^2 \frac{\sigma_1^2}{\sigma_2},$$

and the subscript i should be attached to all the letters when we wish to emphasise that we are dealing solely with the parent population Π_i.

Now we have seen how to test the hypothesis, that the array standard deviations for the populations are the same, and further the additional hypothesis that their regression coefficients are all the same, i.e. that their regression lines are all parallel. To find the probable means of each population, we should have to maximise the only term in which $m_{1,i}$ and $m_{2,i}$ occur, namely,

$$e^{-\frac{1}{2}\frac{(a_i - m_{2,i} + \beta_i m_{1,i})^2}{\sigma^2_{2,i} Q_i / M}}$$

with regard to $m_{1,i}$ and $m_{2,i}$. But these give

$$\beta_i (a_i - m_{2,i} + \beta_i m_{1,i}) = 0 \quad \text{and} \quad a_i - m_{2i} + \beta_i m_{1i} = 0.$$

These, supposing β_i to be finite, reduce to the one equation

$$a_i - m_{2,i} + \beta_i m_{1,i} = 0,$$

and tell us that the position of the mean of the individual population is not fixed, but its most probable position is on the observational regression line of the ith sample. If, on the other hand, we suppose not a series of surfaces with the same σ_1, σ_2, and ρ and variable means, but one surface with one mean m_1, m_2, then we have to minimise

$$\sum_1^k \frac{(a_i - m_2 + \beta_i m_1)^2}{\sigma_2^2 Q_i} = \sum_1^k (X_i)^2 / M.$$

We will put this into another form. Let $\bar{a} = m_2 + \bar{\beta} m_1$, and

$$\sigma_2^2 Q_i = \sigma_1^2 \left\{ \frac{\sigma_2^2}{\sigma_1^2} (1 - \rho^2) + (\beta_i - \bar{\beta})^2 \right\} = \sigma_1^2 \{c^2 + (\beta_i - \bar{\beta})^2\}.$$

Thus we have to minimise

$$\sum_1^k \left(\frac{(a_i - \bar{a} + m_1 (\beta_i - \bar{\beta}))^2}{c^2 + (\beta_i - \bar{\beta})^2} \right)$$

by differentiation with regard to $\bar{a}$ and m_1. We at once obtain the equations

$$\left.\begin{aligned} \bar{a} \sum_1^k \frac{1}{c^2 + (\beta_i - \bar{\beta})^2} - m_1 \sum_1^k \frac{\beta_i - \bar{\beta}}{c^2 + (\beta_i - \bar{\beta})^2} &= \sum_1^k \frac{a_i}{c^2 + (\beta_i - \bar{\beta})^2} \\ \bar{a} \sum_1^k \frac{\beta_i - \bar{\beta}}{c^2 + (\beta_i - \bar{\beta})^2} - m_1 \sum_1^k \frac{(\beta_i - \bar{\beta})^2}{c^2 + (\beta_i - \bar{\beta})^2} &= \sum_1^k \frac{a_i (\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2} \end{aligned}\right\} \quad \text{.....(liv),}$$

to find $\bar{a}$ and m_1, or, of course, m_2 and m_1. Actually we ought to substitute in the above equations the value of c and $\bar{\beta}$ found from differentiating those quantities as they appear not only in Y but in X also. But the resulting equations then involve four variables $\bar{a}$, m_1, c and $\bar{\beta}$ to be solved by approximation, and we will merely ask whether the likely values found for c and $\bar{\beta}$, when the populations have independent means, are compatible with a common mean.

Collecting the data we require, we have

$$c^2 = \cdot 035{,}871, \qquad \bar{\beta} = \cdot 525{,}865, \qquad \sigma_1 = 498 \cdot 201.$$

$$\beta_1 = \cdot 43907, \qquad \beta_2 = \cdot 63677, \qquad \beta_3 = \cdot 55719, \qquad \beta_4 = \cdot 48307.$$

$$\beta_1 - \bar{\beta} = - \cdot 086{,}795, \quad \beta_2 - \bar{\beta} = + \cdot 110{,}905, \quad \beta_3 - \bar{\beta} = + \cdot 031{,}325, \quad \beta_4 - \bar{\beta} = - \cdot 042{,}795.$$

$$(\beta_1 - \bar{\beta})^2 = \cdot 0075{,}3337, \quad (\beta_2 - \bar{\beta})^2 = \cdot 0122{,}9992, \quad (\beta_3 - \bar{\beta})^2 = \cdot 0009{,}8126, \quad (\beta_4 - \bar{\beta})^2 = \cdot 0018{,}3141.$$

Hence

	$(i=1)$	$(i=2)$	$(i=3)$	$(i=4)$
$c^2+(\beta_i-\bar{\beta})^2$	·0434,0437,	·0481,7092,	·0368,5226,	·0377,0241,
$\dfrac{1}{c^2+(\beta_i-\bar{\beta})^2}$	23·039,154,	20·759,413,	27·135,381,	26·523,502,
$\dfrac{\beta_i-\bar{\beta}}{c^2+(\beta_i-\bar{\beta})^2}$	−1·9996,8337,	+2·3023,2270,	+·8500,1581,	−1·1350,7327,
a_i	−259·0895,	−672·8414,	+377·8726,	+1953·1193,
$\dfrac{a_i}{c^2+(\beta_i-\bar{\beta})^2}$	−5969·2029,	−13967·7925,	+10253·7170,	+51803·5637,
$\dfrac{(\beta_i-\bar{\beta})^2}{c^2+(\beta_i-\bar{\beta})^2}$	+·1735,6247,	+·2553,3912,	+·0266,2686,	+·0485,7541,
$\dfrac{a_i(\beta_i-\bar{\beta})}{c^2+(\beta_i-\bar{\beta})^2}$	+518·09696,	−1549·09803,	+321·19768,	−2216·93351.

Thus we obtain the summations, namely,

$$\sum_1^k \frac{1}{c^2+(\beta_i-\bar{\beta})^2}=97{\cdot}457{,}450,\quad \sum_1^k \frac{\beta_i-\bar{\beta}}{c^2+(\beta_i-\bar{\beta})^2}=\cdot0175{,}8187,\quad \sum_1^k \frac{(\beta_i-\bar{\beta})^2}{c^2+(\beta_i-\bar{\beta})^2}=\cdot5041{,}0386,$$

$$\sum_1^k \frac{a_i}{c^2+(\beta_i-\bar{\beta})^2}=+42120{\cdot}2853,\quad \sum_1^k \frac{a_i(\beta_i-\bar{\beta})}{c^2+(\beta_i-\bar{\beta})^2}=-2926{\cdot}73690,$$

and our equations for $\bar{a}$ and m_1 take the forms

$$\left.\begin{aligned}97{\cdot}457{,}450\,\bar{a}-\cdot0175{,}8187\,m_1&=42120{\cdot}2853\\ \cdot0175{,}8187\,\bar{a}-\cdot5041{,}0386\,m_1&=-2926{\cdot}73690\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(\text{lv}).$$

Solving these equations, we find*

$$\bar{a}=433{\cdot}2416,\quad m_1=5820{\cdot}9316,\quad m_2=\bar{a}+m_1\bar{\beta}=3494{\cdot}2658.$$

If we examine the Table on p. 243 these values do not look unreasonable. But the problem is: Will they give reasonable values for X_i? Accordingly we proceed to compute X_i by insertion of $\bar{a}$, m_2, c, $\bar{\beta}$, σ_1 in (xxxiv). We find

$$X_1=-32{\cdot}6340,\quad X_2=-11{\cdot}9121,\quad X_3=+3{\cdot}7550,\quad X_4=+37{\cdot}1555.$$

It is idle to work out P_{λ_m} for four such values supposed to be selected at random from a normal curve of standard deviation unity! The probability is enormously against them.

We conclude confidently that the four sets although they might be samples from a single *shifted* normal population could not possibly be drawn from the same normal population fixed in position. Thus (*b*) of p. 228 is absolutely excluded in this case.

* I have endeavoured in the course of this paper to give some evidence of the nature and amount of arithmetical work required, as this seems to me an essential need for judgment of the value of any statistical process.

We may now proceed to further hypotheses. We may ask, admitting that the four samples could have been drawn from the same normal surface with the values of σ_1, c and $\bar{\beta}$ determined on pp. 245 and 249, shifted parallel to itself, whether the means of these surfaces $m_{1,i}$, $m_{2,i}$ could be reasonably supposed to lie on one and the same straight line

$$y = L + Kx.$$

We have first to determine the most probable value of L and K by minimising

$$\overset{k}{S}_{1}(X_i^2) = \overset{k}{S}_{1} \frac{M(\alpha_i - m_{2,i} + \beta_i m_{1,i})^2}{\sigma_1^2\{c^2 + (\beta_i - \bar{\beta})^2\}} \quad \text{.....(lvi),}$$

where c, $\bar{\beta}$ and σ_1 have the values already determined.

This may be written

$$\overset{k}{S}_{1}(X_i^2) = \overset{k}{S}_{1} \frac{M(\bar{y}_i - \beta_i \bar{x}_i - L + (\beta_i - K)m_{1,i})^2}{\sigma_1^2\{c^2 + (\beta_i - \bar{\beta})^2\}} \quad \text{.....(lvi}^{bis}\text{).}$$

The problem now is: What value shall we give to $m_{1,i}$?

On pp. 147—8 of his memoir, Mr Welch appears to give $m_{1,i}$ the x mean of the ith population the value of the sample mean $\bar{x}_i$. I am a little doubtful if this is admissible. He writes:

"It will be necessary to express algebraically the condition that the population means of the groups [my $m_{1,i}$ and $m_{2,i}$] are collinear. If $\bar{x}_t$ and $\bar{y}_t$ are the means for the n_t observations in the tth group and the symbol [ ] is used to denote expected or population value, then

$$[\bar{y}_t] = \beta_t [\bar{x}_t] + \alpha_t \qquad (t = 1, 2, \ldots k)$$

and the condition that the point $([\bar{x}_t], [\bar{y}_t])$ lies on the line

$$y = lx + m$$

is that

$$\beta_t [\bar{x}_t] + \alpha_t = l[\bar{x}_t] + m \qquad (t = 1, 2, \ldots k)."$$

In my notation $[\bar{y}_t] = m_{2,i}$ and $m_{1,i}$ is not *à priori* $= \bar{x}_i$, my $\bar{x}_i$. We have the condition that

$$m_{2,i} = L + Km_{1,i}.$$

Further, by our (iii),

$$\bar{y}_i = \alpha_i + \beta_i \bar{x}_i,$$

and, if we assume all possible samples taken from the parent population, this leads to

$$m_{2,i} = \alpha_i + \beta_i m_{1,i},$$

and not to

$$m_{2,i} = \alpha_i + \beta_i \bar{x}_i,$$

unless we assume that $m_{1,i} = \bar{x}_i$.

Now the distribution of $\bar{x}_i$ and $\bar{y}_i$ is given by

$$z = z_0 e^{-\frac{1}{2}M\left\{\frac{(\bar{x}_i - m_{1,i})^2}{\sigma_{1,i}^2} - \frac{2\rho(\bar{x}_i - m_{1,i})(\bar{y}_i - m_{2,i})}{\sigma_{1,i}\sigma_{2,i}} + \frac{(\bar{y}_i - m_{2,i})^2}{\sigma_{2,i}^2}\right\}} \quad \text{.....(lvii),}$$

where z_0 does not contain $m_{1,i}$ or $m_{2,i}$. Hence if we maximise the probability of $\bar{x}_i$ and $\bar{y}_i$ occurring—the parent-population means $m_{1,i}$ and $m_{2,i}$ being unknown and not linked together—we find the most likely values of $m_{1,i}$ and $m_{2,i}$ are those of the sample, i.e. $\bar{x}_i$ and $\bar{y}_i$; but if we introduce a bond between $m_{1,i}$ and $m_{2,i}$ this is

not the case. For example, if we make $m_{2,i} = L + Km_{1,i}$ substituting this in (lvii) above, and then minimise

$$\frac{(\bar{x}_i - m_{1,i})^2}{\bar{\sigma}^2_{1,i}} - \frac{2\rho\,(\bar{x}_i - m_{1,i})\,(\bar{y}_i - L - Km_{1,i})}{\bar{\sigma}_{1,i}\,\bar{\sigma}_{2,i}} + \frac{(\bar{y}_i - L - Km_{1,i})^2}{\bar{\sigma}^2_{2,i}}$$

with regard to $m_{1,i}$, this leads to

$$m_{1,i} - \bar{x}_i = \frac{(K - \bar{\beta}_i)\,(\bar{y}_i - L - K\bar{x}_i)}{\frac{\bar{\sigma}^2_{2,i}}{\bar{\sigma}^2_{1,i}}(1 - \rho^2) + (K - \bar{\beta}_i)^2} \quad \text{.....(lvii}^{bis}\text{).}$$

Accordingly $m_{1,i}$ will not be equal to $\bar{x}_i$ unless either the slope of the proposed line of population means is equal to the slope of the ith population regression line (not the regression line of the sample), or the line of population means actually passes through the mean of the ith sample. As these are very special cases, we may assume that in general $m_{1,i}$'s most likely value when it is linked to $m_{2,i}$ is not $\bar{x}_i$.

It is quite true that: (a) the k populations may not be normal populations, although if we feel unable to accept (v) they narrow down to something very close to it, owing to the form (i) of the population surface selected; and (b) the sample may not be a random sample from the ith population—the x-values may be directly selected; still $\bar{x}_i$, $\bar{y}_i$ must correspond to one of the possible samples from the ith population*, and we think (lvii^{bis}) will give a more likely value for $m_{1,i}$ than taking $m_{1,i}$ equal to $\bar{x}_i$.

Now we have seen (p. 252) that

$$\sum_1^k (X_i)^2 = \sum_1^k \frac{M\,(a_i - m_{2,i} + \beta_i m_{1,i})^2}{\bar{\sigma}^2_{1,i}\,\{c_i^2 + (\beta_i - \bar{\beta}_i)^2\}}$$
$$= \sum_1^k \frac{M\,\{\bar{y}_i - L - Kx_i + (\beta_i - K)\,(m_{1,i} - \bar{x}_i)\}^2}{\bar{\sigma}^2_{1,i}\,\{c_i^2 + (\beta_i - \bar{\beta}_i)^2\}}.$$

Or, substituting from (lvii) for the most probable value of $m_{1,i} - \bar{x}_i$, we have, since

$$c_i^2 = \frac{\bar{\sigma}^2_{2,i}}{\bar{\sigma}^2_{1,i}}(1 - \rho_i^2) = \frac{\bar{\gamma}_i^2}{\bar{\sigma}^2_{1,i}},$$

$$\sum_1^k (X_i^2) = \sum_1^k \frac{Mc_i^2}{\bar{\gamma}_i^2} \frac{(\bar{y}_i - L - K\bar{x}_i)^2\,\{c_i^2 + (K - \bar{\beta}_i)\,(\beta_i - \bar{\beta}_i)\}^2}{\{c_i^2 + (\beta_i - \bar{\beta}_i)^2\}\,\{c_2^2 + (\bar{\beta}_i - K)^2\}^2},$$

or, as we may write it,

$$= \sum_1^k \frac{Mc_i^2}{\bar{\gamma}_i^2} \frac{\{\bar{y}_i - \bar{\beta}_i - L - (K - \bar{\beta}_i)\,\bar{x}_i\}^2\,\{c_i^2 + (K - \bar{\beta}_i)\,(\beta_i - \bar{\beta}_i)\}^2}{\{c_i^2 + (\beta_i - \bar{\beta}_i)^2\}\,\{c_i^2 + (K - \bar{\beta}_i)^2\}^2}$$
$$\text{.....(lviii).}$$

The most general problem would be one in which we should maximise both the X_i and Y_i functions for c_i, $\bar{\beta}_i$, L and M. But this leads to extremely complicated equations, and accordingly we will suppose $\bar{\gamma}_i$, c_i and $\bar{\beta}_i$ are the constants determined on pp. 245 and 249, which give reasonable normal surfaces for the

* I think in the actual example chosen, the values of x are more or less a random selection, and accordingly $\bar{x}_i$, $\bar{y}_i$ have random values. But if they are not, and it has been suggested that they might be taken at one end of the surface, is there any logic in assuming that a single probability integral (Welch) of a combination of such integrals, which are interpreted as random measures, will really provide a measure of the possibility of an hypothesis, and not of the selected values of x?

k populations. What we have then to do is to minimise $S(X_i^2)$ with regard to L and $\kappa = K - \bar{\beta}$. Accordingly we start from

$$\sum_1^k S(X_i^2) = \frac{Mc^2}{\bar{\gamma}^2} \sum_1^k S \frac{(\bar{y}_i - \bar{\beta}x_i - L - \kappa\bar{x}_i)^2 \{c^2 + \kappa(\bar{\beta} - \beta_i)\}^2}{(c^2+\kappa^2)^2 \{c^2 + (\bar{\beta} - \beta_i)^2\}}.$$

The differentiation with regard to L gives us, after dividing out the terms independent of i,

$$\sum_1^k S \frac{(\bar{y}_i - \bar{\beta}\bar{x}_i - L - \kappa\bar{x}_i)\{c^2 + \kappa(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} = 0 \quad \text{.....(lix),}$$

and for κ, after again dividing out terms independently of i,

$$\begin{aligned} &- \sum_1^k S \frac{\bar{x}_i (\bar{y}_i - \bar{\beta}x_i - L - \kappa x_i)\{c^2 + \kappa(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} \\ &+ \sum_1^k S \frac{(\bar{y}_i - \bar{\beta}x_i - L - \kappa x_i)^2 \{c^2 + \kappa(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2} \\ &- \frac{2\kappa}{c^2+\kappa^2} \sum_1^k S \frac{(\bar{y}_i - \bar{\beta}\bar{x}_i - L - \kappa\bar{x}_i)^2 \{c^2 + \kappa(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} = 0 \quad \text{.....(lx).} \end{aligned}$$

We will now assume that approximations to L and κ or K are known, and will take $L = L_0 + \xi$, $\kappa = \kappa_0 + \zeta$; thus neglecting squares and products, we have from (lix),

$$\begin{aligned} &\sum_1^k S \frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} = \xi \sum_1^k S \frac{\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta}_i)^2} \\ &\quad + \zeta \left[\sum_1^k S \frac{x_i \{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta}_i)^2} - \sum_1^k S \frac{2(\bar{y}_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2} \right] \\ &\qquad \text{.....(lxi).} \end{aligned}$$

We have now to arrange (lx) in like manner—

$$\begin{aligned} &\sum_1^k S \frac{\bar{x}_i (y_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} - \sum_1^k S \frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2 \{c^2 + \kappa_0(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2} \\ &\quad + \frac{2\kappa_0}{c^2 + \kappa_0^2} \sum_1^k S \frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2 \{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} \\ &= \xi \left[\sum_1^k S \frac{\bar{x}_i \{c^2 + \kappa(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} - 2 \sum_1^k S \frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2 \{c^2 + \kappa_0(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2} \right. \\ &\quad \left. + \frac{4\kappa_0}{c^2 + \kappa_0^2} \sum_1^k S \frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} \right] \\ &+ \zeta \left[\sum_1^k S \frac{\bar{x}_i^2 \{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} - 4 \sum_1^k S \frac{\bar{x}_i (\bar{y}_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2} \right. \\ &\quad + \sum_1^k S \frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2 (\beta_i - \bar{\beta})^2}{c^2 + (\beta_i - \bar{\beta})^2} + \frac{4\kappa_0}{c^2 + \kappa_0^2} \sum_1^k S \frac{x_i (\bar{y}_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} \\ &\quad - \frac{4\kappa_0}{c^2 + \kappa_0^2} \sum_1^k S \frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2 \{c^2 + \kappa_0(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2} \\ &\quad \left. - \frac{2(c^2 - \kappa_0^2)}{(c^2 + \kappa_0^2)^2} \sum_1^k S \frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2 \{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} \right] \qquad \text{.....(lxii).} \end{aligned}$$

The problem is now how to find approximate values of L_0 and $\kappa_0 = K_0 - \bar{\beta}$, which may be inserted in the above equations. Let us see whether Mr Welch's assumption $m_{1,i} = \bar{x}_i$ will serve as a starting point. We have from (lvi)

$$\overset{k}{S}_{1}(X_i^2) = \overset{k}{S}_{1} \frac{M(\alpha_i - m_{2,i} + \beta_i m_{1,i})^2}{\bar{\sigma}^2_{1,i}(c_i^2 + (\beta_i - \bar{\beta}_i)^2)}.$$

Putting $m_{2,i} = L + K m_{1,i} = L + K\bar{x}_i$, $\bar{\sigma}^2_{1,i} = \bar{\gamma}^2/c^2$, $\bar{\beta}_i = \bar{\beta}$ and $c_i = c$, we have

$$\overset{k}{S}_{1}(X_i^2) = \frac{Mc^2}{\bar{\gamma}^2} \overset{k}{S}_{1} \frac{(\bar{y}_i - L - K\bar{x}_i)^2}{c^2 + (\beta_i - \bar{\beta})^2}.$$

Differentiating this result with regard to L and K we have the following relations to determine those quantities after dividing out quantities independent of i:

$$\overset{k}{S}_{1} \frac{(\bar{y}_i - L - K\bar{x}_i)}{c^2 + (\beta_i - \bar{\beta})^2} = 0, \quad \overset{k}{S}_{1} \frac{(\bar{y}_i - L - K\bar{x}_i)\bar{x}_i}{c^2 + (\beta_i - \bar{\beta})^2} = 0.$$

These lead to

$$\left.\begin{aligned} \overset{k}{S}_{1} \frac{\bar{y}_i}{c^2 + (\beta_i - \bar{\beta})^2} &= L \overset{k}{S}_{1} \frac{1}{c^2 + (\beta_i - \bar{\beta})^2} + K \overset{k}{S}_{1} \frac{\bar{x}_i}{c^2 + (\beta_i - \bar{\beta})^2} \\ \overset{k}{S}_{1} \frac{\bar{x}_i \bar{y}_i}{c^2 + (\beta_i - \bar{\beta})^2} &= L \overset{k}{S}_{1} \frac{\bar{x}_i}{c^2 + (\beta_i - \bar{\beta})^2} + K \overset{k}{S}_{1} \frac{\bar{x}_i^2}{c^2 + (\beta_i - \bar{\beta})^2} \end{aligned}\right\} \text{.....(lxiii),}$$

delightfully simple equations after those we have found for a more probable value of $\bar{x}_{1,i}$.

Substituting the values given for $\bar{x}_i$, $\bar{y}_i$ and β_i on p. 243 and for c^2 and $\bar{\beta}$ on p. 252 we find

$$\overset{k}{S}_{1} \frac{\bar{\gamma}_i}{c^2 + (\beta_i - \bar{\beta})^2} = 381{,}806{\cdot}66114, \quad \overset{k}{S}_{1} \frac{1}{c^2 + (\beta_i - \bar{\beta})^2} = 97{\cdot}457{,}450,$$

$$\overset{k}{S}_{1} \frac{\bar{x}_i}{c^2 + (\beta_i - \bar{\beta})^2} = 640{,}114{\cdot}12925, \quad \overset{k}{S}_{1} \frac{\bar{x}_i^2}{c^2 + (\beta_i - \bar{\beta})^2} = 45{,}8190{,}8819{\cdot}95,$$

and

$$\overset{k}{S}_{1} \frac{\bar{x}_i \bar{y}_i}{c^2 + (\beta_i - \bar{\beta})^2} = 28{,}5448{,}6571{\cdot}56.$$

Thus we obtain the equations

$$\left.\begin{aligned} 381{,}806{\cdot}66114 &= 97{\cdot}457{,}450L + 640{,}114{\cdot}12925K \\ 28{,}5448{,}6571{\cdot}56 &= 640{,}114{\cdot}12925L + 45{,}8190{,}8819{\cdot}95K \end{aligned}\right\} \text{.....(lxiv),}$$

which provide the solutions

$$L = -2114{\cdot}2489 \quad \text{and} \quad K = {\cdot}9183{,}6118,$$

and the most likely lines for the means

$$y = {\cdot}9183{,}6118x - 2114{\cdot}2489.$$

We have now to consider what is the probability of the resulting set of X_i's. We have

$$X_i = \frac{\sqrt{M}c}{\bar{\gamma}} \frac{\bar{y}_i - L - K\bar{x}_i}{\sqrt{c^2 + (\beta_i - \bar{\beta})^2}}.$$

Here $M = 8$, $c^2 = \cdot035{,}871$, $\bar{\beta} = \cdot525{,}865$ and $\bar{\gamma} = 94\cdot3575$: see pp. 250 and 245. Thus

$$X_i = \frac{\cdot005{,}67728\,(\bar{y}_i + 2114\cdot2489 - \cdot9183{,}6118\bar{x}_i)}{\sqrt{\cdot035{,}871 + (\beta_i - \cdot525{,}685)^2}},$$

whence we obtain

$$X_1 = +4\cdot7702, \quad X_2 = -6\cdot9129, \quad X_3 = -4\cdot1473, \quad X_4 = 5\cdot8650.$$

These values as a random sample from a normal curve are most highly improbable —we do not need to calculate their P_{λ_n}—and accordingly if we adopt the hypothesis $m_{1,i} = \bar{x}_i$, it is clear that the means of the populations cannot lie on a straight line. Our problem, however, is whether the values thus found, i.e.

$$L_0 = -2114\cdot2489, \quad K_0 = \cdot9183{,}6118,$$

and accordingly $\kappa_0 = K_0 - \bar{\beta} = \cdot3924{,}9618,$

will be suitable approximate values for the solution. The actual arithmetical work is not so laborious as it appears, the summations often repeating themselves. We find (lxi) gives

$$-41\cdot8567{,}47095 = \cdot2035{,}5520\xi + 1458\cdot100{,}055\zeta,$$

and (lxii) $\quad -226{,}852\cdot3567 = -24767\cdot384{,}440\xi + 850{,}2270\cdot6475{,}7242\zeta.$

As solution of these equations we find

$$\xi = -\cdot6633{,}5362, \quad \zeta = -\cdot0286{,}13755 \quad \text{.....(lxv).}$$

These denote a very small change in L, and a fairly small change in K or κ, so we shall not go to a second approximation. They give us

$$L = -2114\cdot9122{,}5362, \quad K = \cdot8897{,}4742 \text{ and } \kappa = \cdot3638{,}8242,$$

or the most likely line for the means of the four populations is

$$y = \cdot8897{,}4742x - 2114\cdot9122{,}5362 \quad \text{.....(lxvi).}$$

We have now to discover whether this line gives a reasonable confirmation of the hypothesis that the four populations means really lie on it. For this purpose we must compute the four X_i's from the formula

$$X_i = \frac{\sqrt{M}c}{\bar{\gamma}} \frac{1}{c^2 + \kappa^2} \frac{(\bar{y}_i - L - M\bar{x}_i)\,\{c^2 + \kappa(\beta_i - \bar{\beta})\}}{\sqrt{c^2 + (\beta_i - \bar{\beta})^2}}$$

$$= \cdot0337{,}36821 \times \frac{(\bar{y}_i + 2114\cdot91225 - \cdot8897{,}4742\bar{x}_i)\,\{\cdot035{,}871 + \cdot3638{,}8242\,(\beta_i - \bar{\beta})\}}{\sqrt{\cdot035{,}871 + (\beta_i - \cdot525{,}865)^2}},$$

and then see if they are a random sample from a normal curve of standard deviation unity. The values of X_i are

$$X_1 = +\cdot191{,}657, \quad X_2 = -1\cdot089{,}254, \quad X_3 = +\cdot573{,}000, \quad X_4 = +1\cdot606{,}296,$$

giving

$p_1 = \cdot575{,}9942$,	$\log p_1 = \bar{1}\cdot760{,}4181$,	$p_1' = \cdot424{,}0058$,	$\log p_1' = \bar{1}\cdot627{,}3719$,
$p_2 = \cdot138{,}0218$,	$\log p_2 = \bar{1}\cdot139{,}9476$,	$p_2' = \cdot861{,}9782$,	$\log p_2' = \bar{1}\cdot935{,}4963$,
$p_3 = \cdot716{,}6757$,	$\log p_3 = \bar{1}\cdot855{,}3227$,	$p_3' = \cdot283{,}3243$,	$\log p_3' = \bar{1}\cdot452{,}2838$,
$p_4 = \cdot945{,}8935$,	$\log p_4 = \bar{1}\cdot975{,}8422$,	$p_4' = \cdot054{,}1065$,	$\log p_4' = \bar{2}\cdot733{,}2494$.

$$\lambda_n = -1\cdot268{,}4694, \quad \lambda_n' = -2\cdot251{,}5986,$$

and hence

$$P_{\lambda_n} = I(1{\cdot}460{,}3793,\ 3) = {\cdot}3351, \quad P_{\lambda_{n'}} = I(2{\cdot}592{,}2486,\ 3) = {\cdot}7598,$$

or

$$Q_{\lambda_n} = {\cdot}6649, \quad Q_{\lambda_{n'}} = {\cdot}2402.$$

$Q_{\lambda_{n'}}$ is the more stringent, but we see that it is quite compatible with the means of the populations being co-linear.

Summary. (*a*) The hypothesis that the four populations are normal and have a common array standard deviation can be accepted. This is opposed to Mr Welch's conclusion that the latter must be rejected, and is curious considering that our assumption of normality is more stringent than his assumption (i). I attribute the difference, supposing the arithmetic of both of us to be correct, to his assuming for the distribution of his compound criterion a curve which is probably not the true curve. In my case the hypothesis of normality leads to the exact curve.

(*b*) I find no improbability in the four normal populations having the same regression coefficient. This is in accordance with Mr Welch's conclusion. This result can be expressed by saying that the four populations have the same values for $\sigma_{2,i}\sqrt{1-\rho^2}$, $\beta = \rho\sigma_{2,i}/\sigma_{1,i}$ and $\sigma_{1,i}$.

(*c*) It is highly improbable that the value of $\bar{a}_i$ is the same for all populations; in particular that they are all selections from a single normal surface.

(*d*) Mr Welch finds that the means of the four populations are unlikely to be collinear. I find that if I use the values he adopts for the $m_{1,i}$'s—namely, to take them equal to the sample means—that I agree with him. But I doubt the accuracy of this equality, and by taking what I think is a more likely value of $m_{1,i}$, I conclude that the hypothesis of collinearity need by no means be rejected.

(*e*) For such small samples as are here under discussion the conclusion that certain hypotheses are not markedly improbable does not seem to be a justification without further experience that they are true. My method has been to combine the probability integrals deduced from the k populations and determine whether they are a fairly probable random sample of k points from the appropriate curve of distribution. Mr Welch's method is to form a combined criterion, and find the probability integral of this combined criterion on the assumption that it follows an empirical curve. In either case with merely four points, we should get equally probable results had we taken other arbitrary curves for my appropriate curves, or for Mr Welch's empirical curve. It requires not four, but frequently one or two hundred points to make a selection markedly more probable when taken from one curve rather than from another. Hence it seems to me unjustifiable to adopt from four samples of eight individuals each any conclusion as to the algebraical forms of the populations sampled.

MISCELLANEA.

(i) Constancy in the Number of Ligulate Flowers of Chrysanthemum Leucanthemum, variety Pinnatifidum, during the Flowering Season.

By WILLIAM DOWELL BATEN

(University of Michigan).

Many plants produce flowers at the beginning of the flowering season which differ significantly from those produced at the end of the season. Yule* found that the average number of sepals in flowers of *Anemone nemorosa* differed significantly for two periods of the same season. Burkill† shows by his counts that the number of stamens in *Stellaria media* varied greatly during the blossoming period. Species of Salpiglossis produce rather large blossoms at the beginning of the season and very tiny cleistogamous blossoms at the close of the season‡. The average numbers of pistils and stamens of *Ranunculus ficaria* differ widely during the time of blooming§. Shull's|| samples of *Aster prenanthoides* Muhl. show that the number of bracts, the number of disk flowers and the number of ligulate flowers differ significantly for different times in the blossoming period. Tower¶ found there was a significant difference between the average number of ray flowers per head of *Chrysanthemum leucanthemum* for different periods of the flowering season.

The object of this study was to determine whether or not *Chrysanthemum leucanthemum*, variety *pinnatifidum*, exhibits any significant differences in the average number of ray flowers per head at different times during the flowering season.

Material.

On May 17, 1932, a sample of one thousand flower heads of *Chrysanthemum leucanthemum*, variety *pinnatifidum*, was gathered at random from a plot of about one-half acre, located about four miles east of Dexter, Michigan, along the Huron River Drive. On this date there were

* Yule, G. U. "Variation of the number of sepals in *Anemone nemorosa*." *Biometrika*, Vol. I. (1901—2), pp. 307—309.

† Burkill, I. H. "On some variations in the number of stamens and corpels." *Journal of the Linnean Society*, Vol. XXXI. (1895), pp. 216—245.

‡ Dale, E. E. Unpublished results of study made in the Botanical Gardens of the University of Michigan.

§ MacLeod, J. "Over de correlatie tusschen het aantal meeddraden en het aantal stampers bij het Speenkruid (*Ficaria ranunculoides*)." *Botanisch Jaarboek*, Jaargang XI. (1899), pp. 91—107. "Cooperative investigations on plants. II. Variations and correlation in Lesser Celandine from divers localities." *Biometrika*, Vol. II. (1902—3), pp. 145—164, Miscellanea. "Change in organic correlation of *Ficaria ranunculoides*," *Biometrika*, Vol. I. (1901—2), pp. 125—128.

|| Shull, G. H. "A quantitative study of variation in the bracts, rays and disk florets of *Aster shortii* Hook., *A. Novae-Angliæ* L., *A. puniceus* L. and *A. prenanthoides* Muhl. from Yellow Springs, Ohio." *American Naturalist*, Vol. XXXVI. (1902), pp. 111—152.

¶ Tower, W. L. "Variation in the ray-flowers of *Chrysanthemum leucanthemum* L. at Yellow Springs, Green County, Ohio, with remarks upon the determination of Modes." *Biometrika*, Vol. I. (1901—2), pp. 309—315.

many heads in full bloom, with comparatively few heads having gone to seed. There were thousands of buds on the plants, showing that this sample was gathered at the beginning of the flowering season.

On June 4, 1932, another sample of one thousand was taken at random from this same plot.

Another sample of one thousand heads was gathered from this plot on June 23, 1932, at which time there were scarcely any buds, while there were thousands of old heads in seed. A few days later there were not enough blooms for another sample. It seems plausible to assume that this sample was gathered at the very end of the flowering season for this particular plot.

The plants were growing very close together in an open place well elevated and well drained. The area was completely covered with daisies with a scattering of uncultivated rye. There was an extraordinary abundance of the species growing here.

The sample taken at the beginning of the season will be designated by Sample A, the one gathered in the middle of the season by Sample B, and the last by Sample C*.

Analysis of the Samples.

The following table gives the frequency distributions of the number of ray flowers per head for the three samples:

Number of ray flowers per head	Sample A Frequency	Sample B Frequency	Sample C Frequency	Sum of samples Frequency
11—12	0	0	3	3
13—14	131	158	139	428
15—16	242	239	241	722
17—18	238	263	240	741
19—20	229	206	232	667
21—22	146	117	128	391
23—24	10	7	17	34
25—26	3	6	0	9
27—28	0	4	0	4
29—30	1	0	0	1
Totals	1000	1000	1000	3000

Histograms in Fig. 1 show how the sample at the beginning of the flowering season compares with the sample at the close of the season. The general appearances of the two are the same while the ranges are nearly the same. Both rise abruptly for the class 13—14 ray flowers, have about the same heights for the three middle classes and drop suddenly for the class 23—24 ray flowers.

The significance of the means is

$$S = \frac{\text{Difference of Means}}{\text{Probable error of Difference of Means}} = 9,$$

which indicates that the difference of the two means is well within fluctuations due to random sampling. Vertical lines near the middle of Fig. 1 represent the means for the two distributions

* The author is indebted to W. H. Long, a student in the Forestry Department of the University of Michigan, for the last two counts.

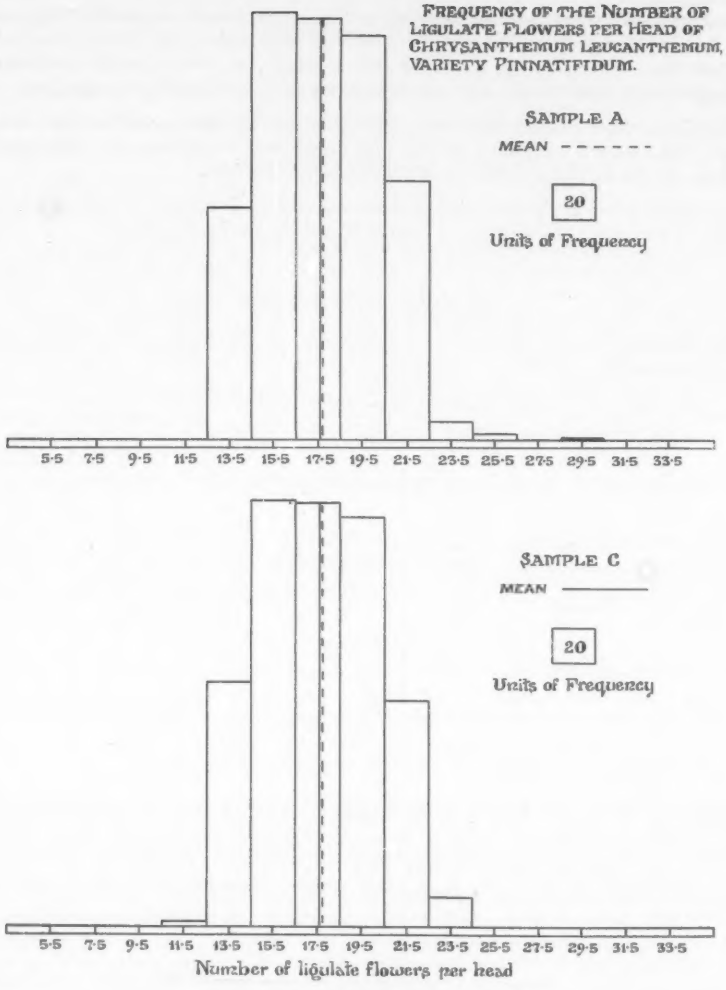

FREQUENCY OF THE NUMBER OF LIGULATE FLOWERS PER HEAD OF CHRYSANTHEMUM LEUCANTHEMUM, VARIETY PINNATIFIDUM.
SAMPLE A
MEAN
20
Units of Frequency
5·5 7·5 9·5 11·5 13·5 15·5 17·5 19·5 21·5 23·5 25·5 27·5 29·5 31·5 33·5
SAMPLE C
MEAN
20
Units of Frequency
5·5 7·5 9·5 11·5 13·5 15·5 17·5 19·5 21·5 23·5 25·5 27·5 29·5 31·5 33·5
Number of ligulate flowers per head

Fig. 1.

and show the small difference between them. The difference between the averages is not significant and is no doubt due to chance. The average number of ray flowers per head did not change appreciably from the beginning of the flowering season to the end.

The following table gives the chief characteristics* of the distributions of the number of ray flowers per head for the three samples, and for the sample made up of the sum of A, B and C.

Sample	Mean	Standard Deviation	Skewness	Median
A	17·630 ray flowers	2·64482 ray flowers	·237819	17·57 ray flowers
B	17·400 ,,	2·6844 ,,	·358392	17·29 ,,
C	17·522 ,,	2·62364 ,,	·108745	17·48 ,,
For sum	17·5167 ,,	2·6052 ,,	·217802	17·438 ,,

The means of the three samples differ by amounts which are less than three probable errors of the various differences of the means and hence are well within fluctuations due to random sampling. This fact clearly indicates the constancy in the average number of ray flowers per head. These means are represented by the vertical lines near the middle of Fig. 2 (p. 264).

Slight differences between the medians of the three samples exhibit also the constancy of the number of ray flowers produced by this species. Their differences are within sampling fluctuations. The standard deviations, which show how closely the items of the distributions are scattered about the means, indicate here the similarity of the three distributions and the small differences in the probable errors of the means. Differences between the standard deviations are not significant and indicate that the samples were no doubt taken at random from the same parent population.

The skewnesses for the distributions do not present any significant differences.

Fig. 2 (p. 264) exhibits frequencies of the samples taken at the beginning of the flowering season, at the middle of the season and at the end. All three distributions contain few heads which had less than 13 ray flowers and a very small percentage which had more than 25. The bulk of all three lies within five classes which have frequencies differing slightly. The central graph differs slightly from the other two but only by differences which are due to fluctuations in sampling from the same parent.

The above study for this species in the composite family clearly shows that the number of ligulate flowers per flower head remained constant throughout the flowering season, and that different climatic conditions which existed during the season did not affect the average number of ray flowers per head.

Analysis of entire Set of Ligulate Flowers.

Since the first three characteristics of the three samples did not differ significantly, a large sample was formed by combining all three distributions. The chief characteristics of this sample for the entire season are given in the table on p. 265. Fig. 3 (p. 265) exhibits the histogram for this distribution.

* The corrections for these characteristics were made according to the formula for correcting the moments for discrete variates. *Annals of Mathematical Statistics*, February 1930, p. 111.

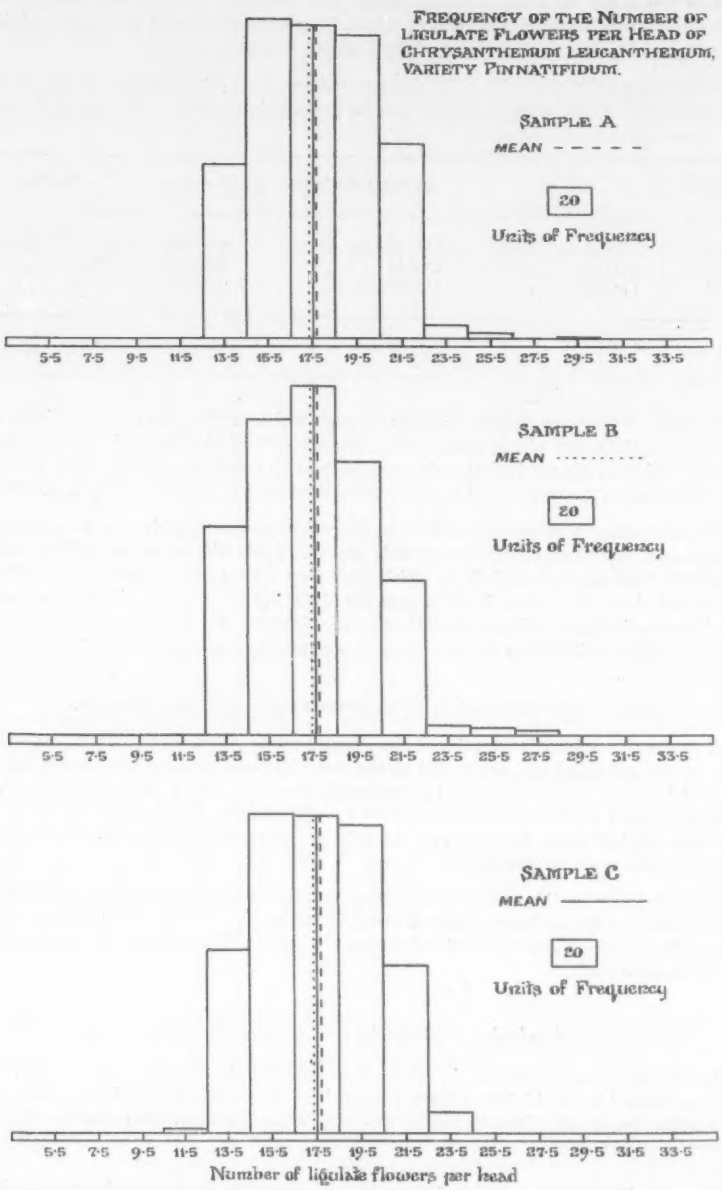

FREQUENCY OF THE NUMBER OF LIGULATE FLOWERS PER HEAD OF CHRYSANTHEMUM LEUCANTHEMUM, VARIETY PINNATIFIDUM.
SAMPLE A
MEAN - - - - -
20
Units of Frequency
5·5 7·5 9·5 11·5 13·5 15·5 17·5 19·5 21·5 23·5 25·5 27·5 29·5 31·5 33·5
SAMPLE B
MEAN .....
20
Units of Frequency
5·5 7·5 9·5 11·5 13·5 15·5 17·5 19·5 21·5 23·5 25·5 27·5 29·5 31·5 33·5
SAMPLE C
MEAN ———
20
Units of Frequency
5·5 7·5 9·5 11·5 13·5 15·5 17·5 19·5 21·5 23·5 25·5 27·5 29·5 31·5 33·5
Number of ligulate flowers per head

Fig. 2.

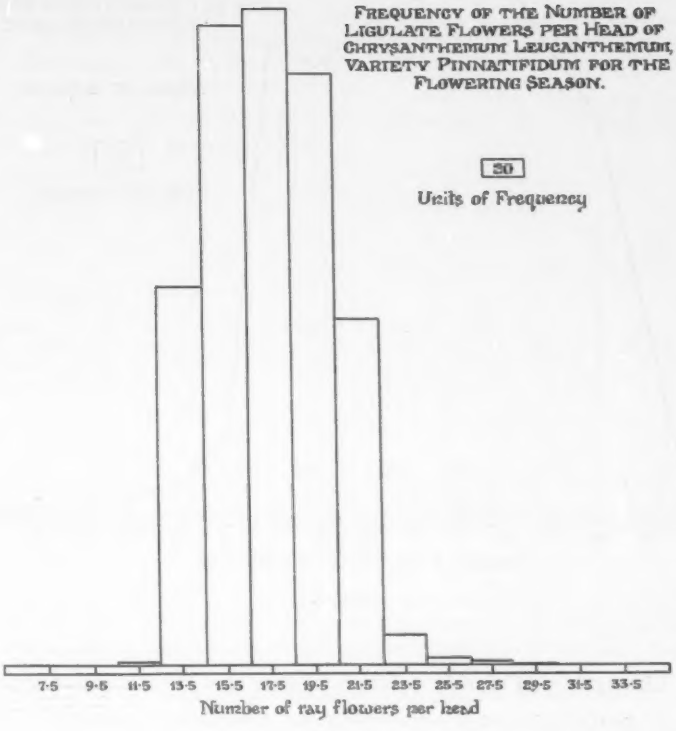

FREQUENCY OF THE NUMBER OF LIGULATE FLOWERS PER HEAD OF CHRYSANTHEMUM LEUCANTHEMUM, VARIETY PINNATIFIDUM FOR THE FLOWERING SEASON.
20
Units of Frequency
7·5 9·5 11·5 13·5 15·5 17·5 19·5 21·5 23·5 25·5 27·5 29·5 31·5 33·5
Number of ray flowers per head

Fig. 3.

The following table gives the frequencies for the data before the grouping was made:

Number of ray flowers	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
Frequency	3	142	286	332	390	374	367	348	319	335	56	22	12	5	4	4	0	1

The frequency polygon in Fig. 4 (p. 266) shows that there are modes at 16 and 21 as Ludwig found in his sample of this species*.

* Ludwig, F. "Ueber Variationskurven und Variationsflächen der Pflanzen." *Botanisches Centralblatt*, Bd. 64 (1895), S. 1—11.

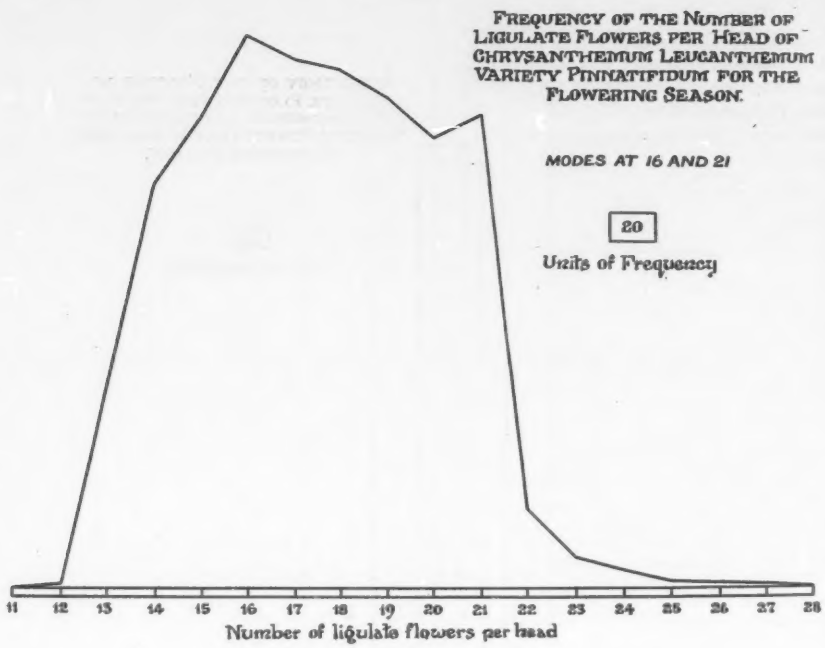

FREQUENCY OF THE NUMBER OF LIGULATE FLOWERS PER HEAD OF CHRYSANTHEMUM LEUCANTHEMUM VARIETY PINNATIFIDUM FOR THE FLOWERING SEASON.
MODES AT 16 AND 21
20
Units of Frequency
11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28
Number of ligulate flowers per head

Fig. 4.

(ii) Strength of Vision in relation to Eye and Hair Pigmentation.

By CATHERINE M. THOMPSON and ETHEL M. ELDERTON

(Written up by K. PEARSON).

Many years ago Francis Galton raised the question* of whether it was not desirable to select candidates for the Civil Service not only by their intellectual powers and the capacity to satisfy a medical examiner, but, especially in the case of tropical appointments, by physical characters which do not enter into the customary medical inquiry. For example, was it more profitable to send young men of light or dark complexion to hot countries? What hair and eye pigmentation were more suited to India or again to Central Africa?

Last year my colleague, Professor Fawcett, asked me with a like end in view whether I knew of any data associating vision with pigmentation. Besides much data not yet reduced taken in the Anthropometric Laboratory at University College, London, I had on loan and in course of reduction data from the Cambridge Anthropometric Laboratory. This latter material offered in the case of some 1500 young men, between 18 and 22 years of age, the means of determining for one visual character, strength of sight, the relation to pigmentation. It did not, however, provide the effect of bright light of various intensities on modifying this relation of sight and pigmentation, and for this reason could scarcely be of service for Professor Fawcett's purpose. The results

* See *The Life, Letters and Labours of Francis Galton*, Vol. II. pp. 386—396, especially p. 395, and Vol. III. p. 220.

appeared of sufficient interest in themselves to deserve publication in the *Miscellanea* of this Journal. The material was extracted from the Cambridge data and tabled by Miss Catherine M. Thompson, a student in the Department of Statistics in University College, London. The constants were then worked out by Dr E. M. Elderton, and partly, under her supervision, by Miss Thompson. The Editor is solely responsible for the arrangement of the material and the conclusions drawn from it. He would state, however, that the Right Eye is known from other investigations to be more long-sighted than the Left.

The visual character taken was the distance at which diamond type could be read without glasses. That distance was measured in inches.

TABLE I. *Strength of Vision and Hair Tint.*

	Red	Fair	Brown	Dark	Jet Black	Total Population
A. Right Eye						
Number...	70	279	355	784	14	1502
Mean Distance... ...	54″·07	55″·19	56″·80	55″·42	53″·57	55″·62
Variability (S. D.) ...	16″·00	15″·36	15″·32	15″·16	14″·22	15″·29
Standard Error of Mean	1″·91	0″·92	0″·81	0″·54	3″·80	0″·39
B. Left Eye						
Number...	69	277	347·5	789·5	14	1497
Mean Distance... ...	51″·92	52″·68	54″·49	53″·91	54″·29	53″·73
Variability (S. D.) ...	15″·60	15″·31	14″·09	14″·66	14″·14	14″·71
Standard Error of Mean	1″·88	0″·92	0″·76	0″·52	3″·78	0″·38

Examining the standard errors, we see that no two eye-strengths are significantly different except in the case of the total populations, where the difference of Left and Right Eye strengths $=1''{\cdot}891$ with a standard error of $0''{\cdot}548$, or the difference is 3·45 times its standard error, which would only occur thrice in 10,000 trials, if Left and Right Eyes were samples of the same normal universe. It would be seen that the Right Eye for each hair-colour is stronger than the Left Eye except for Jet-Black Hairs, of which there were only fourteen. If we take the difference for the Right Eye between Fair and Brown, it is $1''{\cdot}608$, with a standard error of $1''{\cdot}227$. Thus the difference is only 1·31 times its standard error, and would be found to be as great or greater in 9·5 per cent. of trials. If we add all Right and Left Eyes together for Red and Fair, and again for Brown, Dark and Jet Black, we get:

	Mean distance	Variance (σ^2)
Fair and Red Hair ...	53″·752	239·733,774
Brown, Dark and Jet Black	54″·651	220·882,399

Excess Dark Hair $=0''{\cdot}899$.

$$\text{Standard Error of Difference} = \left(\frac{239{\cdot}733{,}774}{695} + \frac{220{\cdot}882{,}399}{2304}\right)^{\frac{1}{2}}$$

$$= 0''{\cdot}6639.$$

Ratio of Difference to Standard Error $= 1{\cdot}354$.

Thus, such an excess in Dark-Hair vision might be expected to occur in about 8·8 per cent. of trials, if no such excess really existed. This is hardly better than the previous test for Right Eye only. No stress can be laid on it. At the same time the general run of stronger sight is the same for both eyes, and we have some little evidence that Red and Fair Hairs have weaker sight than Brown or Dark, although the differences are too small to be significant on the present numbers.

Table II gives the corresponding data for three eye-tints. There are three classifications: Right Eyes, Left Eyes, and both eyes when the Left Eyes have had their strength increased up to Right-Eye strength by the mean difference. Here we naturally find again the same significant difference between the total populations of Right Eyes and Left Eyes. If we consider the difference between the strengths of Light and Dark Right Eyes, it is 4″·607 with the standard error 2″·033, or the difference is 2·27 times its standard error, which would only occur 12 times in 1000 trials, if the Left Eyes and Right Eyes were samples from the same normal universe. On the other hand, comparing Fair and Dark Left Eyes the difference is only 2″·948, with a standard error of 2″·454, or the difference is only 1·20 times its standard error, and would occur in 11·5 per cent. of samples if the Fair and Dark Left Eyes were members of the same universe. The fact is that the Cambridge category "Light Eyes" was either badly defined or poorly distinguished; there are far too few cases in it. (Of course in more modern inquiries eye-scales of 12 to 16 tints are used.)

TABLE II. *Strength of Vision and Eye Tint.*

	Light	Medium	Dark	Total Population
A. Right Eye				
Number	27	992	499	1518
Mean Distance	59″·72	55″·84	55″·11	55″·67
Variability (S. D.) ...	9″·93	15″·24	15″·54	15″·28
Standard Error of Mean	1″·91	0″·48	0″·70	0″·39
B. Left Eye				
Number	27	985	502	1514
Mean Distance	56″·57	53″·77	53″·63	53″·77
Variability (S. D.) ...	12″·32	15″·02	14″·26	14″·73
Standard Error of Mean	2″·37	0″·48	0″·64	0″·38
C. Right Eye + Left reduced				
Number	54	1977	1001	3032
Mean Distance	59″·10	55″·76	55″·32	55″·67
Variability (S. D.) ...	11″·20	15″·30	15″·10	15″·18
Standard Error of Mean	1″·52	0″·34	0″·48	0″·28

It is at once obvious that there is no distinction in strength of vision between the categories of Medium and Dark Eyes. To throw more light on the subject the Left Eyes were modified to Right Eyes as explained above, and now from Class C the difference of strength between Fair and Dark Eyes is 3″·777, with a standard error of 1″·597, which gives the difference 2·36 times its standard error. Such a difference would occur only nine times in 1000 trials if Light and Dark Eyes belonged to the same visual universe, or, there is a very considerable degree of probability that Fair Eyes can see farther than Dark Eyes, i.e. the latter are more short-sighted.

These results throw, of course, very little light on the problem of the complexion suitable to tropical countries. But it is of interest to have the superior vision of the Right Eye confirmed, and further to learn that Light-eyed persons are more far-sighted than Dark-eyed; while on the other hand there is some evidence, but far from conclusive, that Dark-haired persons have stronger vision than Light-haired persons. According to this suggestion the strongest vision would be found in persons with dark hair and blue or grey eyes.

(iii) Note on the Sampling Distribution of $\sqrt{\beta_1}$, where the Population is Normal.

By P. WILLIAMS, B.A.

In a previous issue of this *Journal* Dr E. S. Pearson has published a table containing approximate 5°/₀ and 1°/₀ limits for the distributions of $\sqrt{\beta_1}$ and β_2 when sampling from a normal population*. The results were based on the quadrature of empirical curves (Type VII and Type IV of K. Pearson's system), the values of whose moment coefficients were obtained from expansions in series of inverse powers of n, the sample size, carried as far as terms in n^{-3}. Later work by Professor R. A. Fisher has made it possible to obtain for $\sqrt{\beta_1}$ exact values for the sampling moments up to μ_8†. These may be written as follows: (1), (2) and (3) are given in Dr Pearson's note‡ and (4) in a paper by J. Pepper§.

$$\mu_2(\sqrt{\beta_1}) = \frac{6(n-2)}{(n+1)(n+3)} \quad \text{.....(1),}$$

$$B_2(\sqrt{\beta_1}) = \frac{\mu_4}{\mu_2^2} = 3 + \frac{36(n-7)(n^2+2n-5)}{(n-2)(n+5)(n+7)(n+9)} \quad \text{.....(2),}$$

$$B_4(\sqrt{\beta_1}) = \frac{\mu_6}{\mu_2^3} = 15 + \frac{540\{n^7+60n^6-131n^5-2798n^4-3629n^3+21352n^2+32943n-70070\}}{(n-2)^2(n+5)(n+7)(n+9)(n+11)(n+13)(n+15)} \quad \text{.....(3),}$$

$$B_6(\sqrt{\beta_1}) = \frac{\mu_8}{\mu_2^4}$$

$$= \frac{105(n+1)^3(n+3)^3\{n^6+171n^5+13893n^4+580401n^3-5131014n^2+14132268n-12932920\}}{(n-2)^3(n+5)(n+7).....(n+21)} \quad (4).$$

In a short note Dr Pearson‡ has shown that the approximate expressions he had used for the moments differed only slightly from the true values; at $n=50$: $\sigma_{\sqrt{\beta_1}}$ was correct as far as the fourth decimal place, while the differences in $B_2(\sqrt{\beta_1})$ were as follows:

	$n=50$	$n=75$	$n=100$
True value	3·452	3·351	3·284
Value used in making tables	3·46	3·35	3·28

This disagreement is not large, but it seemed worth while to recalculate the corresponding Type VII curves, and also to make use of Professor Fisher's results to investigate the question of adequacy of approximation a little more fully. In the first place the Type VII curve was refitted for $n=50$ (the smallest value of n considered by Dr Pearson), using the true moment coefficients; the 5°/₀ and 1°/₀ limits were found to be identical, to three decimal places, with those previously given.

The Type VII curves have been used as an approximation to the unknown true distribution; their first four moment coefficients have the correct values‖, but their higher even moment coefficients will of course be in error. The comparison in Table I (p. 270) is of interest.

At $n=50$, B_4 and B_6 for the Type VII curve are so close to the true values that the approximation may be expected to be good. At $n=25$ the difference is especially large for B_8. Since it is likely that a frequency curve with the correct first *eight* moment coefficients would be nearer the true distribution than the Type VII curve, it was decided to make a further com-

* *Biometrika*, Vol. XXII. p. 248. The tables are reprinted in *Tables for Statisticians and Biometricians*, Part II. Table XXXVII^{bis}, p. 224. These limits correspond to the points beyond which the chances are ·05 and ·01 of a sample value falling, and are useful in forming a rapid judgment on the significance of departure from normality.

† *Proc. Roy. Soc.* A, Vol. 130 (1930), pp. 16—22.

‡ *Biometrika*, Vol. XXII. p. 423. § *Biometrika*, Vol. XXIV. p. 60.

‖ Since the Type VII curve is symmetrical, all its odd moment coefficients are necessarily correct.

TABLE I.

Sample size	True values from equations (3) and (4)		Values for Type VII curve with correct $\sigma_{\sqrt{\beta_1}}$ and $B_2(\sqrt{\beta_1})$	
	B_4	B_6	B_4	B_6
25	25·4803	297·0075	26·4371	359·2390
50	23·4203	267·4391	23·3940	269·8547

parison at $n=25$, by using one of the Hansmann system of symmetrical curves*. These curves make use of the moments as far as μ_8.

Taking the numerical values for B_4 and B_6 given in Table I, and also the values

$$\sigma_{\sqrt{\beta_1}} = \cdot 435\ 385, \qquad B_2 = 3{\cdot}578\ 325$$

obtained by putting $n=25$ in equations (1) and (2), it was found on applying Hansmann's rules, that his Type (i) curve is the appropriate one. Its equation takes the form

$$y = y_0 \left\{\frac{1-x^2/q}{1+x^2/p}\right\}^{k'} \qquad \text{.....(5),}$$

where $x=\sqrt{\beta_1}/\sigma_{\sqrt{\beta_1}}$, and

$$\begin{cases} q = 98{\cdot}6732, & p = 9{\cdot}21540, \\ k' = 5{\cdot}37496, & y_0 = 0{\cdot}42492. \end{cases}$$

On applying quadrature to this curve, the 5% and 1% limits for $\sqrt{\beta_1}$ are found to lie at 0·711 and 1·061 respectively. Using the Type VII curve the limits lie at 0·710 and 1·056. The former pair are presumably nearer the true values than the latter, but the difference is of no great importance. In fact as n increases above 25 and B_2 approaches the value of 3†, we may anticipate that the Type VII approximation to the unknown true sampling distribution will steadily improve. Dr Pearson's table has therefore been extended in Table II, using the Type VII approximation, except in the case of $n=25$ where Hansmann's curve has been used.

The interpretation of these limits is as follows. If, for example, a sample of 25 is drawn from a normal population, the chance is ·05 that $\sqrt{\beta_1}$ will lie above ·711; it is also ·05 that $\sqrt{\beta_1}$ will lie below −·711. Similarly the chance is ·01 that $\sqrt{\beta_1}$ will lie above 1·061, and ·01 that it will lie below −1·061. It is clear that for samples of 100 or less the sampling variation in $\sqrt{\beta_1}$ is very considerable so that it will be impossible to detect even large departures from normality in the population. The table however should be useful even if only to emphasise this fact.

In calculating the percentage limits of the Type VII curves it was possible to use the corresponding limits of the "t" distribution as tabled by Professor Fisher‡. This we may do by noting that for the general Type VII curve

$$y = y_0 \left(1+\frac{x^2}{a^2}\right)^{-m} \qquad \text{.....(6),}$$

where

$$a^2 = \frac{2\sigma^2 B_2}{B_2-3}, \qquad m = \frac{5B_2-9}{2(B_2-3)} \qquad \text{.....(7).}$$

* G. H. Hansmann, "On certain Non-normal Symmetrical Frequency Distributions," *Biometrika*, xxvi. pp. 129—95.

† The following table shows the changing values of B_2 as n increases:

n	10	15	20	25	30	40	50	60	70
$B_2(\sqrt{\beta_1})$	3·32	3·52	3·57	3·57	3·56	3·50	3·45	3·41	3·37

‡ *Statistical Methods for Research Workers*, Table IV, at end of work.

TABLE II.

Size of sample	Limits for $\sqrt{\beta_1}$	
	5 °/₀	10 °/₀
25	·711	1·061
30	·661	·982
35	·621	·921
40	·587	·869
45	·558	·825
50	·533	·787
60	·492	·723
70	·459	·673
80	·432	·631
90	·409	·596
100	·389	·567

If in Professor Fisher's notation n is the number of degrees of freedom, then (6) becomes the t distribution on writing

$$m=\tfrac{1}{2}(n+1) \quad \text{.....(8),}$$

$$x/\alpha=t/\sqrt{n} \quad \text{.....(9).}$$

Equations (1) and (2) with (7) give the appropriate values of α and m, the n with which to interpolate in the t tables is given by (8), and the resulting 5 °/₀ and 1 °/₀ values for t when inserted in (9) will give the required probability levels for $x=\sqrt{\beta_1}$.

It is hoped shortly to carry out a similar investigation regarding the sampling distribution of β_2.

(iv) Note on Dr Usher's Paper on Epicanthus (pp. 5—25 of this volume).

By KARL PEARSON.

The interpretation of the valuable material collected by Dr Usher is not easy. In his statistics he has excluded Figs. 24 and 25. Further "continuous" and "discontinuous" inheritance may be interpreted differently by different readers. The terms indeed lose much of their meaning, if we accept the view that epicanthus may disappear with growth; because by predicating a loss we can transfer a pedigree from the discontinuous to the continuous group. Further are we to reckon by "offspring" or by "parents"? Neglecting possible disappearance by growth, the Table on the following page represents the data presented by Dr Usher, and the reader is left to interpret the classification as he pleases.

Tabulated Data of Dr Usher's Epicanthus Paper.

Fig.	Number of Cases			Inheritance					
				Through Male		Through Female		Discontinuous	
	Male	Female	No sex given	Offspring	Fathers	Offspring	Mothers	Offspring	Parent
1	1	3	—	—	—	4	1	—	—
2	1	2	—	—	—	2	1	—	—
3	1	—	2	2	1	—	—	—	—
4	—	—	5	—	—	—	—	?5	?1
5	—	—	4	—	—	—	—	4	1
6	1	4	—	—	—	4	3	—	—
7	20	12	—	25	10	6	3	—	—
8	4	—	—	3	1	—	—	?1	?1
9	2	—	—	1	1	—	—	?1	?1
10	5	1	—	—	—	5	1	—	—
11	—	3	—	—	—	—	—	3	1
12	3	4	—	—	—	2	1	4	1
13	—	3	—	—	—	2	1	1	1
14	4	2	—	—	—	1	1	5	3
15	3	—	—	—	—	—	—	3	2
16	5	5	—	2	1	1	1	6	3
17	3	4	—	4	1	5	2	1	1
18	2	3	—	2	1	1	1	—	—
19	8	8	—	1	1	14	4	—	—
20	1	2	—	—	—	—	—	3	2
21	2	2	—	—	—	—	—	4	1
22	2	5	—	—	—	—	—	7	1
23	4	—	—	—	—	—	—	4	1
24*	—	1	—	—	—	—	—	1	1
25*	—	1	—	—	—	—	—	1	1
26	3	1	—	—	—	—	—	4	1
27	—	7	—	—	—	—	—	7	2
28	1	2	—	—	—	1	1	2	2
29	5	—	—	3	1	—	—	2	1
30	3	2	—	—	—	1	1	4	2
31	—	2	—	—	—	1	1	?1	?1
32	2	1	—	—	—	2	1	—	—
33	—	3	—	—	—	—	—	2	2
34	3	—	—	2	1	—	—	—	—
	89	83	11	45	19	52	24	68 (+?8)	30 (+?4)
	(183)								

* Not included by Dr Usher.

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Conducted on behalf of the Anatomical Society of Great Britain and Ireland by

Professor E. Barclay-Smith; Professor J. P. Hill; Professor J. T. Wilson;
Professor H. H. Woollard; Professor Sir Arthur Keith.

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